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THE BRAUER CHARACTERS AND THE CARTAN MATRIX FOR SL (2 , p)

By

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ABSTRACT

One way to study the representation theory of a group is to get hold of the simple modules. Finding the multiplicities of these simple modules as composition factors of the principal indecomposable modules (PIM) is a step in this way. These multiplicities are the entries of the Cartan matrix.

In this paper, we use the " Orthogonality Relation " (theorem 60.5 of [2]) of the Brauer characters to get the inverse of the Cartan matrix for the finite Chevalley group of type A_1 (SL(2,p)) .

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The Brauer Character.

Let F_p be finite field of p elements. The algebraic closure of F_p will be denoted by K. Let G be the special linear group of type A_1, $(SL(2,K))$, and G_1 will denote the finite Chevalley subgroup of G. Recall that G_1 is the group of all 2x2 matrices with determinant one, and entries from the finite field F_p $(SL(2,p))$.

For each non-negative integer r, let $V(r)$ be the corresponding Weyl-module with highest weight r. This means that, $V(r)$ is the module obtained by reduction modulo p of the corresponding simple module with highest weight r in the zero characteristic. The Weyl module is a module for both G and G_1.

The Brauer character afforded by $V(r)$ will be denoted by φ_r. Let G_1^o be the set of P- regular elements of G_1, and $|G_1|$ be the cardinality of G_1. Define:

$$(\varphi_r, \varphi_{r'}) = \frac{1}{|G_1|} \sum_{x \in G_1^o} \varphi_r(x)\, \overline{\varphi_{r'}(x)} \qquad (1)$$

where $\overline{\varphi_{r'}(x)}$ is the complex conjugate of $\varphi_{r'}(x)$.

So, the Orthogonality Relation reads:

$$(\varphi_r, \varphi_{r'}) = C^{-1} \quad , \; 0 \leqslant r, \; r' \leqslant p-1 \qquad (2)$$

where C is the Cartan matrix.

When applying the relation (2), we first do it for odd primes $(P > 2)$, then to calculate it when $P = 2$.

The Brauer Character for $(2,P), P > 2$.

For each $x \in SL(2,P)$, Let (x) denote the conjugacy class containing x, and let $|(x)|$ denote the cardinality of (x). Put:

$$1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ z=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\ c=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\ d=\begin{bmatrix}1 & 0\\ \nu & 1\end{bmatrix},\ a=\begin{bmatrix}\nu & 0\\ 0 & \nu^{-1}\end{bmatrix}$$

where ν is a generator of the cyclic group F_1^*. The group $SL(2,P)$ contains an element b of order $P+1$.

Then we have (theorem 38.1 on P.498 of [2]) :

x	$\lvert(x)\rvert$	Notes
1	1	P- regular class
z	1	P- regular class
c	$\frac{1}{2}(P^2-1)$	P- singular class
d	$\frac{1}{2}(P^2-1)$	P- singular class
zc	$\frac{1}{2}(P^2-1)$	not P-regular class
zd	$\frac{1}{2}(P^2-1)$	not P-regular class
a^L	$P(P+1)$	P-regular class, $1 \leqslant L \leqslant \frac{P-3}{2}$
b^m	$P(P-1)$	P- regular class , $1 \leqslant m \leqslant \frac{P-1}{2}$

Directly from the above table we get:

$$\lvert G_1\rvert = P^3 - P .$$

The representatives of the P- regular conjugacy classes are $1, z, a^L(1 \leqslant L \leqslant \frac{P-3}{2})$, $b^m(1 \leqslant m \leqslant \frac{P-1}{2})$. Let σ be a primitive $(P-1)\underline{th}$ root of unity, and let ρ be a primitive $(P+1)\underline{th}$ root of unity. Then applying the definition of the Brauer character (see P.588 of [1] , or P.362 of [2] for the definition of this character) we have:

$$\left.\begin{aligned}
&\varphi_r(1) = r + 1, \\
&\varphi_r(z) = (-1)^r (r + 1), \\
&\varphi_r(a^L) = \sum_{i=0}^{r} \sigma^{L(r-2i)} \quad , \quad 1 \leq L \leq \frac{P-3}{2} \quad , \\
&\varphi_r(b^m) = \sum_{i=0}^{r} \rho^{m(r-2i)} \quad , \quad 1 \leq m \leq \frac{P-1}{2} \quad .
\end{aligned}\right\} \quad (3)$$

The relation (1) makes us more interested in the "summation" of the values of φ_r at the elements of G_1^o rather than its value at any particular P- regular element. Denote:

$$y_i = \sum_{k=1}^{\frac{P-3}{2}} \sigma^{k(r-2i)} , \qquad (4)$$

$$z_i = \sum_{k=1}^{\frac{P-1}{2}} \rho^{k(r-2i)} , \qquad (5)$$

So we have:

$$\sum_{L=1}^{(P-3)/2} \varphi_r(a^L) = \sum_{i=0}^{r} y_i , \qquad (6)$$

$$\sum_{m=1}^{(P-1)/2} \varphi_r(b^m) = \sum_{i=0}^{r} z_i , \qquad (7)$$

In order to obtain the last two summations, we have the following four cases:

(1) r is an odd integer .

(2) r is an even integer, $r < (P-1)$.

(3) $r = P-1$

(4) r is an even integer, $(P-1) < r < 2(P-1)$.

Case 1; r is an odd integer.

$$\sum_{L=1}^{(P-3)/2} \varphi_r(a^L) = \sum_{i=0}^{(r-1)/2} y_i + y_{r-i} ,$$

$$y_i + y_{r-i} = \sum_{k=1}^{\frac{P-3}{2}} \sigma^{k(r-2i)} + \sigma^{-k(r-2i)}$$

$$= \frac{\sigma^{(r-2i)\frac{P-1}{2}} - \sigma^{(r-2i)}}{\sigma^{(r-2i)} - 1} - \frac{\sigma^{-(r-2i)\frac{P-3}{2}} - 1}{\sigma^{(r-2i)} - 1}$$

$$= -(1 + \sigma^{(r-2i)\frac{p-1}{2}}) = -1-(-1)^r = 0.$$

Then: $$\sum_{L=1}^{\frac{P-3}{2}} \varphi_r(a^L) = 0.$$

Similarly : $$\sum_{m=1}^{(P-1)/2} \varphi_r(b^m) = 0.$$

Hence:

$$(\varphi_o, \varphi_r) = \frac{1}{P^3 - P} \Big[\varphi_r(1) + \varphi_r(z) + P(P+1) \sum_{L=1}^{\frac{P-3}{2}} \varphi_r(a^L) + P(P-1) \sum_{m=1}^{\frac{P-1}{2}} \varphi_r(b^m) \Big]$$

$$= 0$$

Case 2: r is an even integer , $r < (P - 1)$.

$$\sum_{L=1}^{(P-3)/2} \varphi_r(a^L) = \sum_{i=0}^{r} y_i ,$$

$$y_{r/2} = \frac{P-3}{2} ,$$

$$y_i = \frac{\sigma^{(r-2i)} \left(\sigma^{(r-2i)\frac{P-3}{2}} - 1 \right)}{\sigma^{(r-2i)} - 1} = -1, \quad 0 \leq i \leq r, \ i \neq r/2.$$

Then:

$$\sum_{L=1}^{(P-3)/2} \varphi_r(a^L) = y_{r/2} + \sum_{i=0,\, i\neq r/2}^{r} y_i$$

$$= \frac{P-3}{2} - r .$$

Similarly:

$$\sum \varphi_r(b^m) = \frac{P-1}{2} - r$$

Hence:

$$(\varphi_o, \varphi_r) = \frac{1}{P^3-P}\Big[2(r+1)+P(P+1)\Big(\frac{P-3}{2} - r\Big) +$$

$$+P(P-1)\Big(\frac{P-1}{2} - r\Big)\Big] = \frac{P-2(r+1)}{P} .$$

Case 3: r= P- 1 .

$$y_o = y_{r/2} = y_r = \frac{P-3}{3} \quad ,y_i = -1 \text{ otherwise.}$$

z_i is as in case 2.

Hence:

$$(\varphi_o, \varphi_{P-1}) = \frac{1}{P^3-P}\Big[2P+P(P+1)\Big(3\frac{P-3}{2} - (P-3)\Big)$$

$$+P(P-1)\Big(\frac{P-1}{2} - (P-1)\Big)\Big] = 0$$

Case: 4 r is an even integer, $(P-1) < r < 2(P-1)$.

$$y_{\frac{r-(P-1)}{2}} = y_{r/2} = y_{\frac{r+(P+1)}{2}} = \frac{P-3}{2}$$

$y_i = -1$ otherwise.

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$$z_{\frac{r-(P+1)}{2}} = z_{r/2} = z_{\frac{r+(P+1)}{2}} = \frac{P-1}{2}$$

$z_i = -1$ otherwise.

Hence

$$(\varphi_o, \varphi_r) = \frac{1}{P^3 - P}\left[2(r+1) + P(P+1)\left(3\,\frac{P-3}{2} - (r-2)\right) + \right.$$

$$\left. + P(P-1)\left(3\,\frac{P-1}{2} - (r-2)\right)\right] = \frac{3P - 2(r+1)}{P}\ .$$

So, we have just proved:

<u>Lemma 1.</u>

The value of (φ_o, φ_r) is:

(1) $(\varphi_o, \varphi_r) = 0$, r is an odd integer

(2) $(\varphi_o, \varphi_r) = \frac{P-2(r+1)}{P}$, r is an even integer,

$r < P-1$.

(3) $(\varphi_o, \varphi_{P-1}) = 0$

(4) $(\varphi_o, \varphi_r) = \frac{3P-2(r+1)}{P}$, r is an even integer,

$P-1 < r < 2(P-1)$.

Before our final result, we need another lemma.

<u>Lemma 2:</u>

Assume $r \leqslant r'$, and let $0 \leqslant r$, $r' \leqslant P-1$. Then

$$(\varphi_r, \varphi_{r'}) = \sum_{i=0}^{r} (\varphi_o, \varphi_{r'-r+2i})\ .$$

<u>Proof.</u>

The proof is a direct application of the relations(3).

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So we are now ready for our final result.

Proposition 1.

Assume $P > 2$, and let $r \leqslant r'$, $0 \leqslant r$, $r' \leqslant (P-1)$

Then we have:

(1) $(\varphi_r, \varphi_{r'}) = 0$, $r+r'$ is an odd integer.

(2) $(\varphi_r, \varphi_{r'}) = \dfrac{(r+1)\left(P-2(r'+1)\right)}{P}$, $r+r'$ is an even integer, $r+r' < (P-1)$.

(3) $(\varphi_r, \varphi_{r'}) = \dfrac{r(P-2r')}{P}$, $r+r'=P-1$

(4) $(\varphi_r, \varphi_{r'}) = \dfrac{(4P-r-3r'-1)(-P+r+r'+1)+(P+r-r'-1)(r-r'+1)}{2P}$

$r+r'$ is an even integer, $P-1 < r+r' < 2(P-1)$.

(5) $(\varphi_{P-1}, \varphi_{P-1}) = 1$.

Proof:

All the parts of this proposition are straightforward application of the last two lemmas, except part (5) is proved directly from the fact that V(P-1) is the Steinberg module (§2.2 of [3]).

Application for P=7.

$$(\varphi_r, \varphi_{r'}) = 1/7 \begin{bmatrix} 5 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 6 & 0 & -2 & 0 & -3 & 0 \\ 1 & 0 & 3 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 3 & 0 & 1 & 0 \\ -3 & 0 & -2 & 0 & 6 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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From the above matrix,we get its inverse which is the Cartan matrix C for SL(2,7).

$$C=\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix agrees with the result of Jeyakumar in [4] . In this paper the projective cover modules properties are used .

The Brauerr Character for SL(2,2).

Let 1,c,a be as before.SL (2,2) contains an element b of order P+1 .Then we have:

x	$\|(x)\|$	Notes
1	1	P- regular class
c	P^2-1	P- singular class
b	$P(P-1)$	P- regular class

and we get:

$$(\varphi_r, \varphi_{r'}) = \frac{1}{P^3-P}\left[\varphi_r(1)\,\varphi_{r'}(1)+P(P-1)\,\varphi_r(b)\,\varphi_{r'}(b^{-1})\right.$$

Hence

$$(\varphi_r, \varphi_{r'}) = \frac{1}{6}\left[(r+1)(r'+1)(r'+1)+2\,\frac{\sin(r+1)\theta\sin(r'+1)\theta}{\sin^2\theta}\right.$$

where $\theta = 120°$.

Then

$$\left[(\varphi_r, \varphi_{r'})\right] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

REFERENCES

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[3] J.E.Humphreys,Ordinary and Modular Representations of Chevalley Groups,Lecture Notes in Mathematics,528, Springer,1976.

[4] A.V.Jeyakumar,Principal indecomposable representations for the group SL(2,q),J.Algebra 30 ,(1974),444-458.