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1 DEFORMATION OF SURFACES IN RIEMANNIAN 5-SPACES

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4 ABSTRACT

Let $M: D \longrightarrow V^5$ and $\bar{M}: D \longrightarrow \bar{V}^5$ $(D \subset R^2)$ be two surfaces in the second order deformation in the Riemannian 5-spaces V^5 and $\bar{V}^5$ of curvatures R, $\bar{R}$ respectively.

Let $L: T_m(V^5) \longrightarrow T_m(\bar{V}^5)$ be an isometry such that $L(\frac{dm}{dt}) = \frac{d\bar{m}}{dt}$,

$\pi: T_m(V^5) \longrightarrow n = \{v_5\}$. Then M and $\bar{M}$ are in the third order deformation provided that:

1- The Gaussian curvature K and the curvature k of the normal bundle satisfies $K^2 - k^2 \neq 0$ on M.

2- $\dim T_m^2(M) = 4$ on M.

3- M has no non-trivial real conjugate directions at each of its points

4- $L\{R(x,y)z\} = \bar{R}(Lx,Ly)Lz$, and $L\{\pi R(x,y)u\} = \bar{\pi}\bar{R}(Lx,Ly)Lu$ for each $x,y,z \in T_m(M)$, $u \in N_m(M) = \{v_3, v_4\}$

5- M and $\bar{M}$ are in the third order deformation on ∂D.

INTRODUCTION

The conditions for two isometric surfaces to be in the second order deformation was discussed by A. Švec [3]. His result concerns two infinitesimal isometric surfaces. He obtained the conditions for two infinitesimal isometric surfaces in E^3 to have the same second fundamental forms, that is the variation of the second fundamental form $\delta II = 0$.

Talaat [4] generalized Švec's result from the infinitesimal isometric surfaces to the more general isometric surfaces in the Riemannian 3-spaces.

The aim of this paper is to find the conditions for two surfaces in Riemannian 5-spaces in the second order deformation to be in the third order deformation.

DEFORMATION OF SURFACES

Let V^5 be a 5-dimensional Riemannian space, $D \subset R^2$ be a bounded domain. Suppose $M: D \rightarrow V^5$ is a surface such that at each point $m \in M$ the dimension of the osculating space $T_m^2(M)$ is equal to 4. To each $m \in M$, associate a frame such that $\{m, v_1, v_2\}$ is the tangent plane $T_m(M)$ and $\{m, v_1, \ldots, v_4\}$ is the osculating space . Hence we can write

$$\left.\begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2 , && \omega^3 = \omega^4 = \omega^5 = 0, \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , && \omega_1^5 = 0, \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , && \omega_2^5 = 0. \end{aligned}\right\} \tag{1}$$

By exterior differentiation of $\omega^3 = \omega^4 = 0$, and applying Cartan's lemma we get

$$\left.\begin{aligned} \omega_1^3 &= a^3\omega^1 + b^3\omega^2 , && \omega_1^4 = a^4\omega^1 + b^4\omega^2 , \\ \omega_2^3 &= b^3\omega^1 + C^3\omega^2 , && \omega_2^4 = b^4\omega^1 + C^4\omega^2 , \end{aligned}\right\} \tag{2}$$

From (1) and (2) we see that

$$dv_1 = \omega_1^2 v_2 + \omega^1 (a^3 v_3 + a^4 v_4) + \omega^2 (b^3 v_3 + b^4 v_4) ,$$

$$dv_2 = -\omega_1^2 v_1 + \omega^1 (b^3 v_3 + b^4 v_4) + \omega^2 (C^3 v_3 + C^4 v_4).$$

The osculating space $T_m^2(M)$ at each point $m \in M$ is generated by the vectors

$$v_1, v_2, a^3 v_3 + a^4 v_4, b^3 v_3 + b^4 v_4, C^3 v_3 + C^4 v_4. \tag{3}$$

Hence
$$\dim T_m^2(M) = 4 \Longleftrightarrow \text{Rank} \begin{pmatrix} a^3 & b^3 & C^3 \\ a^4 & b^4 & C^4 \end{pmatrix} = 2. \tag{4}$$

Let V,W be two tangent vector fields on M such that

$$V = x\, v_1 + y\, v_2 , \qquad W = \xi v_1 + \eta v_2 , \tag{5}$$

From (1), (2) and (5)

$$\begin{aligned} WV_m = W(x\, v_1 + y v_2) &= Wx\, v_1 + Wy\, v_2 + x\, W v_1 + y\, W v_2 \\ &= Wx\, v_1 + Wy\, v_2 + x\, dv_1(W) + dv_2(W), \end{aligned}$$

From which we get

$$\left.\begin{aligned} WV_m &= \left\{ W_x - y\omega_1^2(W) \right\} v_1 + \left\{ W_y + x\omega_1^2(W) \right\} v_2 + \\ &+ (a^3 x\xi + b^3 x\eta + b^3 y\xi + C^3 y\eta) v_3 + (a^4 x\xi + b^4 x\eta + b^4 y\xi + C^4 y\eta) v_4, \end{aligned}\right\} \quad (6)$$

which shows that the vector $WV_m \in T_m^2(M)$.

Definition: The vector field V is called conjugate if there is another tangent vector field $W \neq 0$, such that WV is a tangent vector field. Equation (6) leads to the following lemma

Lemma: The tangent vector field $V = xv_1 + yv_2$ is conjugate iff

$$(a^3b^4 - a^4b^3)x^2 + (a^3c^4 - a^4c^3)xy + (b^3c^4 - b^4c^3)y^2 = 0 . \qquad (7)$$

Let $\bar{M}: D \longrightarrow \bar{V}^5$ be another surface isometric to M . If $L: T_m(V^5) \longrightarrow T_{\bar{m}}(\bar{V}^5)$ is the isometry, we may choose the frames of M and $\bar{M}$ in such a way that

$$Lm = \bar{m} , \quad L v_i = \bar{v}_i \quad \text{and} \quad L\, dm = d\bar{m} , \qquad (8)$$

Then

$$\omega^1 = \bar{\omega}^1 , \quad \omega^2 = \bar{\omega}^2 , \quad \omega_1^2 = \bar{\omega}_1^2 . \qquad (9)$$

From (9)

$$\left.\begin{aligned} d\bar{m} &= \omega^1 \bar{v}_1 + \omega^2 \bar{v}_2 , \quad \bar{\omega}^3 = \bar{\omega}^4 = \bar{\omega}^5 = 0 , \\ d\bar{v}_1 &= \omega_1^2 \bar{v}_2 + \bar{\omega}_1^3 \bar{v}_3 + \bar{\omega}_1^4 \bar{v}_4 , \quad \bar{\omega}_1^5 = 0, \\ d\bar{v}_2 &= -\omega_1^2 \bar{v}_1 + \bar{\omega}_2^3 \bar{v}_3 + \bar{\omega}_2^4 \bar{v}_4 , \quad \bar{\omega}_2^5 = 0. \end{aligned}\right\} \quad (10)$$

Taking a general curve $\gamma = \gamma(u,v)$, $u = u(t)$, $v = v(t)$ in D, then the corresponding curves on M and $\bar{M}$ are given respectively by

$$M(\gamma(t)) = m(u(t) , v(t)) , \quad \bar{M}(\gamma(t)) = \bar{m}(u(t), v(t)),$$

Therefore

$$\left.\begin{aligned} \frac{dm}{dt} &= m_u u' + m_v v' \\ \frac{d^2m}{dt^2} &= m_{uu}(u')^2 + 2m_{uv} u'v' + m_{vv}(v')^2 + m_u u'' + m_v v'' \end{aligned}\right\} \quad (11)$$

Let

$$\omega^k = A^k du + B^k dv , \quad \omega_i^j = C_i^j du + D_i^j dv , \quad K=1,2,\ i=1,..,4\ j=1,..,5 .$$

Then

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$$dm = m_u du + m_v dv = \omega^1 v_1 + \omega^2 v_2 = (A^1 v_1 + A^2 v_2) du + (B^1 v_1 + B^2 v_2) dv,$$

which gives

$$m_u = A^1 v_1 + A^2 v_2 \quad , \quad m_v = B^1 v_1 + B^2 v_2. \tag{12}$$

From (1) and (12)

$$\left.\begin{aligned}
m_{uu} &= A^1_u v_1 + A^2_u v_2 + A^1(C^2_1 v_2 + C^3_1 v_3 + C^4_1 v_4) + A^2(-C^2_1 v_1 + C^3_2 v_3 + C^4_2 v_4), \\
m_{uv} &= A^1_v v_1 + A^2_v v_2 + A^1(D^2_1 v_2 + D^3_1 v_3 + D^4_1 v_4) + A^2(-D^2_1 v_1 + D^3_2 v_3 + D^4_2 v_4), \\
&= B^1_u v_1 + B^2_u v_2 + B^1(C^2_1 v_2 + C^3_1 v_3 + C^4_1 v_4) + B^2(-C^2_1 v_1 + C^3_2 v_3 + C^4_2 v_4), \\
m_{vv} &= B^1_v v_1 + B^2_v v_2 + B^1(D^2_1 v_2 + D^3_1 v_3 + D^4_1 v_4) + B^2(-D^2_1 v_1 + D^3_2 v_3 + D^4_2 v_4).
\end{aligned}\right\} \tag{13}$$

For the second surface $\bar{M}$ and its curve $\bar{M}(\gamma(t))$, we get by the same procedure $\frac{d\bar{m}}{dt}, \frac{d^2\bar{m}}{dt^2}$,

where:

$$\bar{\omega}^k = \omega^k = A^k du + B^k dv \quad , \bar{\omega}^j_i = \bar{C}^j_i du + \bar{D}^j_i dv \quad , \quad K=1,2, i=1,\dots,4, j=2,\dots,5.$$

Now, introducing

$$\psi^k = L\left(\frac{d^k m}{dt^k}\right) - \frac{d^k \bar{m}}{dt^k}, \tag{14}$$

we give the following definition

<u>Definition</u>: The two surfaces M and $\bar{M}$ are said to be in $k^{\underline{th}}$ order deformation; if and only if $\psi^k = 0$.

Let the two surfaces M and $\bar{M}$ be in a second order deformation, hence

$\psi^2 = L\left(\frac{d^2 m}{dt^2}\right) - \frac{d^2\bar{m}}{dt^2} = 0$, from which we obtain,

$$\sum_{i=1}^{2} A^i(C^j_i - \bar{C}^j_i) = 0 \;, \; \sum_{i=1}^{2} A^i(D^j_i - \bar{D}^j_i) = 0, \sum_{i=1}^{2} B^i(D^j_i - \bar{D}^j_i) = 0, \quad j=3,4. \tag{15}$$

Since the two tangents m_u and m_v are independent, we get from (12)

$$\begin{vmatrix} A^1 & A^2 \\ B^1 & B^2 \end{vmatrix} \neq 0. \tag{16}$$

From $(15_{2,3})$ and (16)

$$D_i^j = \bar{D}_i^j . \qquad i=1,2 , \quad j=3,4. \tag{17}$$

From (17) and comparing the coefficients of v_3 and v_4 of $(13_{2,3})$ for the surfaces M and $\bar{M}$ we get

$$\sum_{i=1}^{2} B^i (C_i^j - \bar{C}_i^j) = 0, \qquad j = 3,4 \tag{18}$$

Equations (15_1), (16) and (18) imply that

$$C_i^j = \bar{C}_i^j . \quad i=1,2 \quad , \quad j = 3,4. \tag{19}$$

Equations (17) and (19) imply that

$$\omega_i^j = \bar{\omega}_i^j \qquad i=1,2 \quad , \quad j = 3,4. \tag{20}$$

Since the osculating space $T_m^2(M)$ at each point $m \in M$ is generated by the vectors

$$v_1 , v_2 , C_1^3 v_3 + C_1^4 v_4 , D_1^3 v_3 + D_1^4 v_4 , \; C_2^3 v_3 + C_2^4 v_4 , \; D_2^3 v_3 + D_2^4 v_4 , \tag{21}$$

then from (21)

$$\dim T_m^2(M) = 4 \Longleftrightarrow \text{Rank} \begin{pmatrix} C_1^3 & C_2^3 & D_1^3 & D_2^3 \\ C_1^4 & C_2^4 & D_1^4 & D_2^4 \end{pmatrix} = 2 \tag{22}$$

From (11)

$$\frac{d^3 m}{dt^3} = m_{uuu} (u')^3 + 3m_{uuv}(u')^3 v' + 3m_{uvv} u'(v')^2 + m_{vvv}(v')^3 + 3m_{uu} u' u'' +$$
$$+ 3m_{uv}(u'v'' + u''v') + 3m_{vv} v'v'' + m_u u''' + m_v v''' . \tag{23}$$

Differentiating each equation of (13) with respect to u,v respectively, considering that the two surfaces M and $\bar{M}$ are in a third order deformation, and equating the coefficients of $\bar{v}_3$, $\bar{v}_4$ and $\bar{v}_5$ in

$$\psi^3 = L\left(\frac{d^3 m}{dt^3}\right) - \frac{d^3 \bar{m}}{dt^3} = 0, \text{ we get}$$

$$\left.\begin{aligned}
&D_i^j(C_3^4-\bar{C}_3^4)=0, \quad C_i^j(C_3^4-\bar{C}_3^4)=0, \quad C_i^j(D_3^4-\bar{D}_3^4)=0, \quad D_i^j(D_3^4-\bar{D}_3^4)=0, \quad i=1,2, j=3,4.\\
&C_i^3(C_3^5-\bar{C}_3^5)+C_i^4(C_4^5-\bar{C}_4^5)=0, \quad D_i^3(C_3^5-\bar{C}_3^5)+D_i^4(C_4^5-\bar{C}_4^5)=0,\\
&C_i^3(D_3^5-\bar{D}_3^5)+C_i^4(D_3^5-\bar{D}_3^5)=0, \quad D_i^3(D_3^5-\bar{D}_3^5)+D_i^4(D_4^5-\bar{D}_4^5)=0.
\end{aligned}\right\} \tag{24}$$

From (22) it follows that

$$\omega_i^j = \bar{\omega}_i^j \qquad i=3,4, \quad j=4,5. \tag{25}$$

Now, let $M \subset V^5$ and $\bar{M} \subset \bar{V}^5$ be two surfaces in a second order deformation. Then the structure equations of M and $\bar{M}$ are given respectively by:

$$\left.\begin{aligned}
&d\omega_i^j = \omega_i^k \wedge \omega_k^j - \frac{1}{2} R_{ik\ell}^j\, \omega^k \wedge \omega^\ell, \quad R_{ik\ell}^j + R_{i\ell k}^j = 0,\\
&d\bar{\omega}_i^j = \bar{\omega}_i^k \wedge \bar{\omega}_k^j - \frac{1}{2} \bar{R}_{ik\ell}^j\, \bar{\omega}^k \wedge \bar{\omega}^\ell, \quad \bar{R}_{ik\ell}^j + \bar{R}_{i\ell k}^j = 0.
\end{aligned}\right\} \tag{26}$$

Let

$$\bar{\omega}_i^j = \omega_i^j + \tau_i^j . \tag{27}$$

From (9), (20), (26) and (27) we get

$$\left.\begin{aligned}
&\omega_1^4 \wedge \tau_3^4 = (R_{112}^3 - \bar{R}_{112}^3)\,\omega^1 \wedge \omega^2, \quad \omega_1^3 \wedge \tau_3^4 = (\bar{R}_{112}^4 - R_{112}^4)\,\omega^1 \wedge \omega^2,\\
&\omega_1^3 \wedge \tau_3^5 + \omega_1^4 \wedge \tau_4^5 = (\bar{R}_{112}^5 - R_{112}^5)\,\omega^1 \wedge \omega^2, \quad \omega_2^4 \wedge \tau_3^4 = (R_{212}^3 - \bar{R}_{212}^3)\,\omega^1 \wedge \omega^2,\\
&\omega_2^3 \wedge \tau_3^4 = (\bar{R}_{212}^4 - R_{212}^4)\,\omega^1 \wedge \omega^2, \quad \omega_2^3 \wedge \tau_3^5 + \omega_2^4 \wedge \tau_4^5 = (\bar{R}_{212}^5 - R_{212}^5)\,\omega^1 \wedge \omega^2,\\
&d\tau_3^4 = -\omega_3^5 \wedge \tau_4^5 - \tau_3^5 \wedge \omega_4^5 - \tau_3^5 \wedge \tau_4^5 + (R_{312}^4 - \bar{R}_{312}^4)\,\omega^1 \wedge \omega^2,\\
&d\tau_3^5 = \omega_3^4 \wedge \tau_4^5 + \tau_3^4 \wedge \omega_4^5 + \tau_3^4 \wedge \tau_4^5 + (R_{312}^5 - \bar{R}_{312}^5)\,\omega^1 \wedge \omega^2,\\
&d\tau_4^5 = -\omega_3^4 \wedge \tau_3^5 - \tau_3^4 \wedge \omega_3^5 - \tau_3^4 \wedge \tau_3^5 + (R_{412}^5 - \bar{R}_{412}^5)\,\omega^1 \wedge \omega^2.
\end{aligned}\right] \tag{28}$$

Now let the three vectors $x, y, z \in T_m(M)$ be such that

$$x = x^1 v_1 + x^2 v_2, \quad y = y^1 v_1 + y^2 v_2, \quad z = z^1 v_1 + z^2 v_2, \quad (x^3 = y^3 = z^3 = 0).$$

Then

$$\left.\begin{aligned} R(x,y)z &= R^{\ell}_{ijk}\, x^j y^k z^i v_\ell \quad , \qquad R^{\ell}_{ijk}+R^{i}_{\ell jk}=0 \\ &= x^1y^2z^1\Big(\sum_1^5 R^{\ell}_{112}v_\ell\Big)+x^2y^1z^1\Big(\sum_1^5 R^{\ell}_{121}v_\ell\Big)+ \\ &\quad +x^1y^2z^2\Big(\sum_1^5 R^{\ell}_{212}v_\ell\Big)+x^2y^1z^2\Big(\sum_1^5 R^{\ell}_{221}v_\ell\Big). \end{aligned}\right\} \quad (29)$$

Again let $u \in N_m(M)$, where $N_m(M) = \{v_3, v_4\}$;

then

$$\begin{aligned} R(x,y)u &= R^{\ell}_{ijk}x^j y^k u^i v_\ell \\ &= (R^{\ell}_{312}x^1y^2u^3+R^{\ell}_{412}x^1y^2u^4+R^{\ell}_{321}x^2y^1u^3+R^{\ell}_{421}x^2y^1u^4)v_\ell \quad . \end{aligned} \quad (30)$$

Let $\pi : T_m(V^5) \longrightarrow n = \{v_5\}$,i.e. $\pi(\sum_1^5 z^i v_i) = z^5 v_5$ is the natural projection, then $\pi R(x,y)u = (R^5_{312}u^3+R^5_{412}u^4)(x^1y^2-x^2y^1)v_5$.

Assume that for every $x,y,z \in T_m(M)$ and $u \in N_m(M)$

$$L\left[R(x,y)z\right] = \bar{R}(Lx,Ly)Lz, \quad L\left[\pi R(x,y)u\right] = \bar{\pi}\,\bar{R}(Lx,Ly)Lu. \quad (31)$$

From (29) $\longrightarrow$ (31) and comparing the coefficients of $\bar{v}_1,\ldots,\bar{v}_5$ we get,

$$\left.\begin{aligned} (R^2_{112}-\bar{R}^2_{112})(x^1y^2-x^2y^1)z^2 &= 0, \\ (R^2_{112}-\bar{R}^2_{112})(x^1y^2-x^2y^1)z^1 &= 0, \\ (x^1y^2-x^2y^1)\left\{(R^3_{112}-\bar{R}^3_{112})z^1+(R^3_{212}-\bar{R}^3_{212})z^2\right\} &= 0, \\ (x^1y^2-x^2y^1)\left\{(R^4_{112}-\bar{R}^4_{112})z^1+(R^4_{212}-\bar{R}^4_{212})z^2\right\} &= 0, \\ (x^1y^2-x^2y^1)\left\{(R^5_{112}-\bar{R}^5_{112})z^1+(R^5_{212}-\bar{R}^5_{212})z^2\right\} &= 0, \end{aligned}\right\} \quad (32)$$

which are valid for every $x^1, x^2, y^1, y^2, z^1, z^2$.

From (32) we get

$$R^{\ell}_{ijk} = \bar{R}^{\ell}_{ijk} \qquad i=1,\ldots,4,\ j,k=1,2, \qquad \ell = 1,\ldots,5. \quad (33)$$

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From (28) and (33)

$$\left.\begin{aligned}
&\omega_1^4\wedge\tau_3^4 = 0\ , \quad \omega_1^3\wedge\tau_3^4 = 0,\\
&\omega_1^3\wedge\tau_3^5+\omega_1^4\wedge\tau_4^5 = 0\ , \quad \omega_2^4\wedge\tau_3^4 = 0,\\
&\omega_2^3\wedge\tau_3^5+\omega_2^4\wedge\tau_4^5 = 0\ , \quad \omega_2^3\wedge\tau_3^4 = 0,\\
&d\tau_3^4=-\omega_3^5\wedge\tau_4^5-\tau_3^5\wedge\omega_4^5-\tau_3^5\wedge\tau_4^5+(R^4_{312}-\bar{R}^4_{312})\,\omega^1\wedge\omega^2,\\
&d\tau_3^5=\omega_3^4\wedge\tau_4^5+\tau_3^4\wedge\omega_4^5+\tau_3^4\wedge\tau_4^5,\\
&d\tau_4^5=-\omega_3^4\wedge\tau_3^5-\tau_3^4\wedge\omega_3^5-\tau_3^4\wedge\tau_3^5.
\end{aligned}\right\}\tag{34}$$

Let us write

$$\tau_3^4 = A\omega^1 + B\omega^2\ ,\ \tau_4^5 = z_1\omega^1+z_2\omega^2,\ \tau_3^5 = y_1\omega^1+y_2\omega^2. \tag{35}$$

From (2), (34) and (35_1)

$$\begin{aligned}
&a^3B - b^3A = 0\ , \quad a^4B - b^4A = 0,\\
&b^3B - c^3A = 0\ , \quad b^4B - c^4A = 0
\end{aligned}\tag{36}$$

From (4) and (36)

$$A = B = 0 \Longrightarrow \tau_3^4 = 0. \tag{37}$$

Thus equation (34) reduces to

$$\left.\begin{aligned}
&\omega_1^3\wedge\tau_3^5+\omega_1^4\wedge\tau_4^5 = 0\ ,\ \omega_2^3\wedge\tau_3^5+\omega_2^4\wedge\tau_4^5 = 0\ ,\\
&\omega_3^5\wedge\tau_4^5+\tau_3^5\wedge\omega_4^5+\tau_3^5\wedge\tau_4^5 = (R^4_{312}-\bar{R}^4_{312})\,\omega^1\wedge\omega^2,\\
&d\tau_3^5=\omega_3^4\wedge\tau_4^5\ , \qquad d\tau_4^5 = -\omega_3^4\wedge\tau_3^5\ .
\end{aligned}\right\}\tag{38}$$

From (2), $(35_{2,3})$ and $(38_{1,2})$

$$a^4z_2-b^4z_1+a^3y_2-b^3y_1=0 \quad , \quad b^4z_2-c^4z_1+b^3y_2-c^3y_1=0 \tag{39}$$

From $(35_{2,3})$, $(38_{4,5})$ and Cartan's lemma, there exist functions $S_1,\dots,S_6$ such that,

$$\left.\begin{aligned} &dy_1-y_2\omega_1^2-z_1\omega_3^4 = S_1\omega^1+S_2\omega^2 \ , \quad dz_1-z_2\omega_1^2+y_1\omega_3^4= S_4\omega^1+S_5\omega^2,\\ &dy_2+y_1\omega_1^2-z_2\omega_3^4 = S_2\omega^1+S_3\omega^2 \ , \quad dz_2+z_1\omega_1^2+y_2\omega_3^4= S_5\omega^1+S_6\omega^2. \end{aligned}\right\} \tag{40}$$

From (2) and Cartan's lemma, there exist functions $\alpha_1,\dots,\alpha_8$ such that,

$$\left.\begin{aligned} &da^3-2b^3\omega_1^2-a^4\omega_3^4= \alpha_1\omega^1+(\alpha_2-\tfrac{1}{2}R^3_{112})\omega^2,\\ &db^3+(a^3-c^3)\omega_1^2-b^4\omega_3^4=(\alpha_2+\tfrac{1}{2}R^3_{112})\omega^1+(\alpha_3-\tfrac{1}{2}R^3_{212})\omega^2,\\ &dc^3+2b^3\omega_1^2-c^4\omega_3^4=(\alpha_3+\tfrac{1}{2}R^3_{212})\omega^1+\alpha_4\omega^2 \ ,\\ &da^4-2b^4\omega_1^2+a^3\omega_3^4=\alpha_5\omega^1+(\alpha_6-\tfrac{1}{2}R^4_{112})\omega^2,\\ &db^4+(a^4-c^4)\omega_1^2+b^3\omega_3^4 = (\alpha_6+\tfrac{1}{2}R^4_{112})\omega^1+(\alpha_7-\tfrac{1}{2}R^4_{212})\omega^2,\\ &dc^4+2b^4\omega_1^2+c^3\omega_3^4 = (\alpha_7+\tfrac{1}{2}R^4_{212})\omega^1+\alpha_8\omega^2. \end{aligned}\right\} \tag{41}$$

Differentiating each equation of (39) and substituting from (40) and (41) we get two homogeneous equations in ω^1 and ω^2, and hence

$$\begin{aligned} b^3S_1-a^3S_2+b^4S_4-a^4S_5 &= -(\alpha_2+\tfrac{1}{2}R^3_{112})y_1+\alpha_1y_2-(\alpha_6+\tfrac{1}{2}R^4_{112})z_1+\alpha_5z_2,\\ b^3S_2-a^3S_3+b^4S_5-a^4S_6 &= -(\alpha_3-\tfrac{1}{2}R^3_{212})y_1+(\alpha_2-\tfrac{1}{2}R^3_{112})y_2-(\alpha_7-\tfrac{1}{2}R^4_{212})z_1+\\ &\quad +(\alpha_6-\tfrac{1}{2}R^4_{112})z_2,\\ c^3S_1-b^3S_2+c^4S_4-b^4S_5 &= -(\alpha_3+\tfrac{1}{2}R^3_{212})y_1+(\alpha_2+\tfrac{1}{2}R^3_{112})y_2-(\alpha_7+\tfrac{1}{2}R^4_{212})z_1+\\ &\quad +(\alpha_6+\tfrac{1}{2}R^4_{112})z_2, \end{aligned} \tag{42}$$

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. . .

$$c^3 S_2 - b^3 S_3 + c^4 S_5 - b^4 S_6 = -\alpha_4 y_1 + (\alpha_3 - \tfrac{1}{2} R^3_{212}) y_2 - \alpha_8 z_1 + (\alpha_7 - \tfrac{1}{2} R^4_{212}) z_2. \tag{42}$$

On the surface M the Gaussian curvature K and the curvature of the normal bundle k are given respectively by :

$$K = a^3 c^3 - (b^3)^2 + a^4 c^4 - (b^4)^2 \quad , \quad k = b^3 (c^4 - a^4) + b^4 (a^3 - c^3). \tag{43}$$

Assuming $K \neq k$ and taking (43) into consideration, certain combinations of (39) lead to,

$$\left.\begin{aligned}
(K-k) y_1 &= (b^3 b^4 - a^3 c^4)(y_2 + z_1) + \left[a^3 b^4 - a^4 b^3 - a^4 c^4 + (b^4)^2 \right] (z_2 - y_1), \\
(K-k) y_2 &= \left[a^4 c^4 - (b^4)^2 + c^3 b^4 - b^3 c^4 \right] (y_2 + z_1) + (b^3 b^4 - c^3 a^4)(z_2 - y_1), \\
(K-k) z_1 &= \left[a^4 b^3 - (b^3)^2 - a^3 b^4 + a^3 c^3 \right] (y_2 + z_1) + (a^4 c^3 - b^3 b^4)(z_2 - y_1), \\
(K-k) z_2 &= (b^3 b^4 - a^3 c^4)(y_2 + z_1) + \left[a^3 c^3 + b^4 c^3 - (b^3)^2 - b^3 c^4 \right] (z_2 - y_1).
\end{aligned}\right\} \tag{44}$$

Certain combinations of $(42_{1,3})$ lead to,

$$\begin{aligned}
(K-k) S_2 &= \left[a^4 c^4 - (b^4)^2 + c^3 b^4 - b^3 c^4 \right] (S_2 + S_4) + (b^3 b^4 - c^3 a^4)(S_5 - S_1) + L_1 (y_2 + z_1) + \\
&\quad + L_2 (z_2 - y_1), \\
(K-k) S_5 &= (b^3 b^4 - a^3 c^4)(S_2 + S_4) + \left[a^3 c^3 + b^4 c^3 - (b^3)^2 - b^3 c^4 \right] (S_5 - S_1) + L_3 (y_2 + z_1) + \\
&\quad + L_4 (z_2 - y_1).
\end{aligned} \tag{45}$$

Certain combinations of $(42_{2,4})$ lead to,

$$\begin{aligned}
(K-k) S_2 &= (b^3 b^4 - a^3 c^4)(S_3 + S_5) + \left[a^3 b^4 - a^4 b^3 - a^4 c^4 + (b^4)^2 \right] (S_6 - S_2) + L_5 (y_2 + z_1) + \\
&\quad + L_6 (z_2 - y_1), \\
(K-k) S_5 &= \left[a^4 b^3 - (b^3)^2 - a^3 b^4 + a^3 c^3 \right] (S_3 + S_5) + (a^4 c^3 - b^3 b^4)(S_6 - S_2) + L_7 (y_2 + z_1) + \\
&\quad + L_8 (z_2 - y_1).
\end{aligned} \tag{46}$$

From (40)

$$\left.\begin{aligned} d(y_2+z_1)+(y_1-z_2)(\omega_1^2+\omega_3^4) &= (S_2+S_4)\omega^1+(S_3+S_5)\omega^2, \\ d(y_1-z_2)-(y_2+z_1)(\omega_1^2+\omega_3^4) &= (S_1-S_5)\omega^1+(S_2-S_6)\omega^2. \end{aligned}\right\} \quad (47)$$

__Theorem:__ Let $M:D \longrightarrow V^5$ and $\bar{M}:D \longrightarrow \bar{V}^5$ be two surfaces in the second order deformation, where $D \subset R^2$ is a bounded domain and V^5, $\bar{V}^5$ are two Riemannian spaces. Suppose that :

1- $K^2 - k^2 \neq 0$ on M , 2- $\dim T_m^2(M) = 4$ on M,

3- M has no non-trivial real conjugate directions at each its points,

4- $L\left[R(x,y)z\right] = \bar{R}(Lx,Ly)Lz$, $L\left[\pi R(x,y)u\right] = \bar{\pi}\bar{R}(Lx,Ly)Lu$ for each $x,y,z \in T_m(M)$, $u \in N_m(M)$,

5- M and $\bar{M}$ are in the third order deformation on the boundary ∂M.

Then M and $\bar{M}$ are in the third order deformation.

__Proof:__ We can choose in the bounded domain $D \subset R^2$ coordinates u,v such that

$$\omega^1 = r\,du \quad , \quad \omega^2 = r\,dv \quad , \quad r=r(u,v). \tag{48}$$

$$\omega_1^2 = F_1\omega^1 + F_2\omega^2 \quad , \quad \omega_3^4 = F_3\omega^1 + F_4\omega^2 \tag{49}$$

From (45_1), (46_1) and (47); similarly from (45_2), (46_2) and (47)

$$\left.\begin{aligned} &(b^3b^4-c^3a^4)\frac{\partial(y_1-z_2)}{\partial u} + \left[a^4c^4-(b^4)^2+a^4b^3-a^3b^4\right]\frac{\partial(y_1-z_2)}{\partial v} + \\ &\quad + \left[(b^4)^2-a^4c^4+b^3c^4-c^3b^4\right]\frac{\partial(y_2+z_1)}{\partial u} + (b^3b^4-a^3c^4)\frac{\partial(Y_2+z_1)}{\partial v} \\ &\quad = f_1(y_1-z_2)+f_2(y_2+z_1), \\ &\left[a^3c^3+b^4c^3-(b^3)^2-b^3c^4\right]\frac{\partial(y_1-z_2)}{\partial u} + (b^3b^4-a^4c^3)\frac{\partial(y_1-z_2)}{\partial v} + \\ &\quad + (a^3c^4-b^3b^4)\frac{\partial(y_2+z_1)}{\partial u} + \left[a^3c^3-(b^3)^2+a^4b^3-a^3b^4\right]\frac{\partial(y_2+z_1)}{\partial v} = \\ &\quad = f_3(y_1-z_2) + f_4(y_2+z_1) \end{aligned}\right] \quad (50)$$

The associated quadratic form λ of (50) is

$$\lambda = (k-K)\left[(a^3 b^4 - a^4 b^3)\mu^2 + (a^3 c^4 - a^4 c^3)\mu\nu + (b^3 c^4 - b^4 c^3)\nu^2 \right], \qquad (51)$$

λ is **definite** because of our assumption that $K^2 - k^2 \neq 0$ and that the surface M has no non-trivial real conjugate directions. Hence the system (50) is elliptic.

Now, since M and $\bar{M}$ are in third order deformation at the boundary ∂M, hence from (25) and (27) we have $\tau_3^5 = \tau_4^5 = 0$ on the boundary ∂M. Then from (50) and (44) $y_1 = y_2 = z_1 = z_2 = 0$ for every $m \in M$. Hence M and $\bar{M}$ are in the third order deformation which proves the theorem.

Q.E.D.

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