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THE STRUCTURE OF BIFURCATIONS OF CRITICAL POINTS OF A DIFFERENTIAL EQUATION WITH SEVERE NON-LINEARITY .

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ABSTRACT

The critical points of an autonomous differential equation of the second order with severe non-linearity , can have a bifurcation structure in a parameter plane (a,b) , $-1<b<1$, similar to the " box-within-a-box " bifurcation structure . This is shown using a recurrence relation having such a bifurcation structure in (a,b) plane , $-1<b<1$.

INTRODUCTION

Consider the recurrence relation T , with real variables , defined by :

$$(1)\qquad \begin{cases} x_{n+1} = f_1(x_n\ ,\ y_n) = 1+y_n-ax_n^2 \\ y_{n+1} = g_1(x_n\ ,\ y_n) = bx_n \end{cases} \quad ,\ n=0\ ,\ x_n=x_0\ ,\ y_n=y_0$$

where a,b are real parameters . The transformation T can be written in vector form as follows :

$$X_{n+1} = F_1(X_n)\ ,\ X = \begin{pmatrix} x \\ y \end{pmatrix}\ ,\ F_1(X) = \begin{pmatrix} f_1(X) \\ g_1(X) \end{pmatrix}\ .$$

By successive applications of T , we obtain the transformation T^k,defined by :

$$(2)\qquad X_{n+k} = F_k(X_n) = F_1(X_{n+k-1})\ ,\ k=1,2,\ldots\ ,\ T^1=T\ .$$

A fixed point of (1) , (cycle of (1) of order 1) , is a point satisfying the equation :

$$(3)\qquad X_{n+1} = X_n = F_1(X_n)\ .$$

A cycle of (1) of order k is a fixed point of T^k which is not a fixed point of T^p , where p is a positive integer less than k . The k points of a cycle of (1) of order k verify the relation :

(4) $X_{n+k} = X_n = F_k(X_n)$.

T has a "box-within-a-box" bifurcation structure ([2]-[7]) in a parameter plane (a,b) , $-1 < b < 1$.

We are now gowing to make use of this structure of bifurcations to find the bifurcation structure of the critical points of a second order differential equation .

THE STRUCTURE OF BIFURCATIONS OF CRITICAL POINTS OF A DIFFERENTIAL EQUATION OF THE SECOND ORDER .

In this section we will try to find a differential equation of the second order , each critical point of which is a point of a cycle of (1) of some order β belonging to the box Ω^j_β, ($X_{j.\ell.\beta}$,$j \in \{1,2,\dots,p_\beta\}$,$\ell \in \{1,2,\dots,\beta\}$) , where the stability of $X_{j.\ell.\beta}$ as a critical point of the required differential equation corresponds to the stability of $X_{j.\ell.\beta}$ as a point of that cycle of (1) of order β. If this happens for all integers β then the structure of bifurcations of the critical points of this differential equation will be similar to the "box-within-a-box" bifurcation structure of (1) .

THEOREM. Any critical point of the differential equation :

(5) $X' = G_k(X) = F_k(X) - X$, $'=d/dt$, $X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$,

is a point of a cycle of (1) of one of the orders $\alpha_{km}=k/m$,$m \in \{i/i$ is a factor of $k\}$, and vice versa . For $|b|< 1$, the stability correspondence is complete for m even , and is incomplete for m odd .

We give an illustration for the case k=1 . For k=1 , the fixed points of (1) , ($X_{j.1.1}$) , are those points X satisfying the equation (3). But equation (3) is the equation giving the critical points of (5) with k=1 . Hence the set of fixed points of (1) equals the set of critical points of (5) with k=1 . Let $s_{j.1.1}$ be an eigenvalue of $X_{j.1.1}$ as a fixed point of (1) , (each point $X_{j.\ell.\beta}$ has two eigenvalues $s_{j.\gamma.\beta}$, γ=1,2) ,then $s_{j.1.1}$ is an eigenvalue of the matrix $F_{1x}(X_{j.1.1})$, where $F_{1x}(X)$ is the matrix with columns $(\partial F_1/\partial x_i)(X)$, i=1,2 , $x_1=x$,$x_2=y$. Let $\lambda_{j.1.1}$ be the corresponding eigenvalue of $X_{j.1.1}$, as a critical point of (5) . Then $\lambda_{j.1.1}$ is an eigenvalue of the matrix $G_{1x}(X_{j.1.1})$. It is easy to see that the following relation is satisfied :

Since $X_{j.1.1}$ as a fixed point of (1) is stable when $|s_{j.1.1}|<1$, while $X_{j.1.1}$ as a critical point of (5) is stable when real part of $\lambda_{j.1.1}<0$, then the last relation shows that the stability is preserved except when $s_{j.1.1}<-1$.

PROOF OF THEOREM.

The set of critical points of (5) are the points X satisfying the equation (4) . But equation (4) is the equation giving the set of points of cycles of (1) of orders $\alpha_{km}=k/m$, where m is a factor of k . Hence the two sets are equal . Let $s_{j.\gamma.\alpha_{km}}$ be an eigenvalue of $X_{j.\ell.\alpha_{km}}$ as a point of a cycle of (1) of order α_{km} , that is , $s_{j.\gamma.\alpha_{km}}$ is an eigenvalue of the matrix $F_{\alpha_{km}x}(X_{j.\ell.\alpha_{km}})$. Let $\lambda_{j.\gamma.\alpha_{km}}$ be the corresponding eigenvalue of $X_{j.\ell.\alpha_{km}}$ as a critical point of (5) , that is , $\lambda_{j.\gamma.\alpha_{km}}$ is an eigenvalue of the matrix $G_{kx}(X_{j.\ell.\alpha_{km}})$. Since $T^k = (T^{\alpha_{km}})^m$, and using the relation between $G_k(X)$ and $F_k(X)$, we have the following relation :

$$(6) \qquad \lambda_{j.\gamma.\alpha_{km}} = s^m_{j.\gamma.\alpha_{km}} - 1 .$$

This relation shows that the stability of $X_{j.\ell.\alpha_{km}}$, $|b|<1$ as a critical point of (5) , corresponds to the stability of $X_{j.\ell.\alpha_{km}}$, as a point of a cycle of (1) of order α_{km} , except for $s_{j.\gamma.\alpha_{km}} < -1$ when m is odd .Hence for m even , there is complete stability correspondence , while for m odd there is incomplete stability correspondence .

COROLLARY .

If we take k=n! and as n increases , the number of cycles of (1) of order α_{km} with even m , corresponding to critical points of (5) , increases . Hence for k=n! and as n increases , the structure of bifurcations of the critical points of (5) approaches a structure similar to the "box-within-a-box" bifurcation structure in (a,b) plane , $|b|<1$ .The following table gives , for k= 1! , 2! , ... ,6! , the order α_{km} of cycles with complete stability correspondence , order α_{km} of cycles with incomplete stability correspondence .

n	k=n!	α_{km} with even m	α_{km} with odd m
1	1	--	1
2	2	1	2
3	6	1 , 3	2 , 6
4	24	1,2,3,4,6,12	24 , 8
5	120	1,2,3,4,5,6,10,12,15,20,30,60	8,24,40,120
6	720	1,2,3,4,5,6,8,9,10,12,15,18,20, 24,30,36,40,45,60,72,90,120,180, 360 .	16,48,80,144,240,720

One of the differences between the two structures of bifurcations is that when $s_{j.\gamma.\alpha_{km}}=1$, two cycles of (1) of order α_{km} appear simultaneously , one of them is stable , the other is unstable , while for the differential equation (5) , $2\alpha_{km}$ critical points of (5) appear simultaneously , α_{km} points from them are stable , the rest are unstable . Another difference is that when $s_{j.\gamma.\alpha_{km}}=-1$, a stable cycle of (1) of order α_{km} becomes unstable and a stable cycle of order $2\alpha_{km}$ appears , while for the differential equation (5) , α_{km} stable critical points of (5) become unstable and $2\alpha_{km}$ stable critical points of (5) appear . It is to be noted that the correspondence between points of cycles of (1) and critical points of (5) from stability point of view , does not guarantees that the two points will have the same type of singularity . For example , it is possible to find a cycle of (1) of order α_{km} with complex eigenvalues which corresponds to α_{km} critical points of (5) with real eigenvalues .For $|b|>1$, it is possible to find stable critical points of (5) , that corresponds to unstable cycle of (1) with complex eigenvalues , but this is not important for us since we are interested only in $|b|<1$.

GENERALIZATION OF THE PREVIOUS RESULT.

A generalization of the previous result is possible using the results stated in [6] . Consider the recurrence relation , with real variable x :

(7) $$x_{n+1} = h_1(x_n,a) \quad , \quad a_1 \leqslant a \leqslant a_2$$

where $h_1(x,a)$ satisfies the following hypotheses :

(i) $h_1(x,a) \in c^r$, $r \geqslant 1$, has with respect to x one extremum only (maximum or minimum) , and is continuous with respect to a ;

(ii) on each side of the extremum A , $h_1(x,a)$ consists of a monotonic increasing arc and a monotonic decreasing arc , both with sufficient regularity . $h_1(x,a)$ has at maximum one inflexion point ; to the right of A , if A is a maximum , or to the left of A , if A is a minimum ;

(iii) as a varies in a monotonic way in the interval (a_1,a_2) , there is a fixed point q_2, whose eigenvalue after being positive takes negative values , with monotonically increasing absolute value . The difference between the ordinate of q_2 , on $x_{n+1} = x_n$ and that of the extremum increases in the same manner .

$h_1(x,a)$ satisfies also one of the following supplementary hypotheses :

(iv) as a varies in a monotonic way in the interval (a_1,a_2) , there exists a fixed point q_1 at finite distance , with positive eigenvalue and absolute value greater than one , for every a ;

(v) as a varies in the interval (a_1,a_2) , the fixed point q_1 is at infinity for every a ;

(vi) as a varies in the interval (a_1,a_2) , the fixed point q_1 exists for certain values of a and does not exist for the other values of a .

The interval (a_1,a_2) is supposed also to be with sufficient length . The recurrence relation (7) , such that $h_1(x,a)$ satisfies (i),(ii),(iii) , and one of the hypotheses (iv),(v),(vi) , has a bifurcations structure "box-within-a-box" similar to that of the recurrence $x_{n+1} = 1-ax_n^2$. In special cases , however , the values a_1^* or $a_{(1)_0}$ are not defined . An example of $h_1(x,a)$ is the function $h_1(x,a) = \exp(a(1-x/k))$ [6] .

Now consider the recurrence relation :

(8) $$X_{n+1} = H_1(X_n) = \begin{pmatrix} h_1(x_n,a) + y_n \\ bx_n \end{pmatrix} \quad , \; X= \begin{pmatrix} x \\ y \end{pmatrix} , |b| < 1$$

The continuous passage of the properties of the case b=o , to that with b≠o takes place [6] .Results similar to that obtained for the differential equation (5) , can be also obtained for the differential equation :

$$X' = L_k(X) - X$$

where H_k is obtained by k successive applications of H_1 .

CONCLUSION

This paper gives , in a simple and easy way , the structure of bifurcations of the critical points of a differential equation of the second order with severe non-linearity , which is very difficult to be known by ordinary methods . It gives us also an idea that the special types of singularities which appear in the bifurcation curves of the recurrence relation (1) can also appear in the bifurcation curves of the critical points of a second order differential equation with the increasing of the degree of nonlinearity .

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