MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

VIBRATIONAL BEHAVIOR OF CRACKED JEFFCOTT ROTORS

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ABSTRACT

A simplified mathematical model has been developed based on Jeffcott rotor to investigate the dynamic behavior of cracked shaft. Analysis has indicated the existance of unstable zones of operation at the vicinity of which rotor response is largely magnified. In general rotor response due to unbalance forces contain vibrations at 1/rev and 2/rev components while gravity excited vibration is characterized by 3/rev harmonics as well. The measured response curve of the rotor shows an increased shaft deflection at all speeds. Also a subcritical peak is detected. Comparing the analytical results with experimental measurements indicates that the mathematical model is very realistic.

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INTRODUCTION

Rotor cracking is one problem which if undetected, could lead to catastrophic failures. With modern high speed a large specific output machines the use of high stresses components became necessary. The probability of crach growth resulting from stress concentrations has been increasing, Recently several turbine - generator rotor have been reported to experience deep cracks, consequently, detection of these cracks has been the subject of several research workers.

Crack growth can be detected by a number of well established techniques, generally grouped together under the label of"non-destructive testing". These techniques include surface inspection methods such as magnetic particle, eddy current or dye penetration and full depth methods employing ultrasonics and radiography.

Unfortunately none of these technique is suitable for on-load inspection with the rotor running, hence it is necessary to search for some other technique for monitoring rotor integrity. Vibration monitoring at the bearing housing is on such technique which is showing some success.

Inorder to develop a sound technique to detect rotor crack growth using vibration measurements, it is essential to study the vibration behavior of cracked rotor. This has been the subject of several publications aimed at studying the changes in dynamic response of a rotor as a consequence of crack onset.

Simplified models representing the case of cracked rotors have been studied by several authors (1-3). The model consisted of a linear spning mass system with viscous damping. The effect of the tangentially running crack was introduced as a lack of bending stiffness crack direction (1-2). Reference (3) assumed an additional flexibility superimposed on shaft flexibility. This model explained qualitatively the response of cracked shafts. Stability of the cracked rotor was discussed by references (2,3). Their conclusions have stated that a cracked rotor was found to behave very similarly to rotors with unequal stiffness. This asymmetry in stiffness is generated as the growth of the crack weakness the rotor in crack direction.

Few authors have quantitatively studied the dynamic response of cracked shafts. Vibrational response to excitation caused by gravity and unbalance has been reported by (4-5). Reference (4) analyzed the steady state response and natural vibrations of general rotor bearing system with open crack using the conventional iterative numerical scheme of the transfer matrix method. The dependence of once/rev and twice/rev vibration components on unbalance phase referred to crack direction was indicated numerically.

Mays and Davies (5) calculated numerically the vibrational

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a standard finite element routine. The method employs the technique of successive approximations and utilizes the fact that the fractional change in rotor stiffness is small even for large cracks. The calculation were compared with experimental results. Reviewing the above mentioned publications the following remarks may be obtained.

a) The qualitative study made by reference 1,2 and 3 did not present defined expressions for both unbalance and gravity responses. These expression were difficult to obtain because of the complicated crack model used by Ref.(2). Derived relations required to study the cracked shaft behavior is rather complicated.

b) Numerically obtained solution 4,5 lack the capability to accurately investigate rotor instability.

The main objective of this work is twofold. First, mathematical model representing the rotating cracked shaft is analyzed and closed form solutions for free and forced vibrations are derived. Second, experimental program is carried out to check the validity of the mathematical model. Analyzing both the analytical and experimental results would help diagnoze cracked induced problems.

ANALYTICAL WORK

In the following, an analytical formulation of the problem is worked out. The behavior of the Jeffcott rotor with a simplified crack modeled is used as described in the following

CRACK MODEL

Figure (1) shows qualitively how the stiffness of the shaft changes with the crack position. Depending on whether the crack is in the compression of the tension zone the shaft is more or less stiff. These stiffness changes are caused by the open-close condition of the crack with shaft rotation. The figure illustrates two types of crack model which are described as:

a - Ideal crack-open-close condition of crack is stepwise (A case studied by reference).

b - Real crack-open-close of crack is continious.

In the subsequent analysis a real crack shown in figure (1.b) is considered. During lateral vibrations of the rotor, the rotor stiffness is small, linear analysis can be employed usefully.

The periodic variation of stiffness with the rotation of the rotation of the rotor can be assumed to vary sinusoidaly. The instantaneous value of stiffness in crack direction (Ku) can be

. . .

$$K_u = K_o - \Delta K \ (1 + \cos wt).$$

where ΔK is the lack of stiffness due to crack. Hence,

$K_u = K_1 - \Delta K \cos wt$, and $k_v = k_o$. Then,

$$w_n^2 = k_o/M \quad \text{and} \quad w_1^2 = k_1/M \qquad (0)$$

DESCRIPTION OF THE ROTOR MODEL

The rotor is modeled as a two degree-of-freedom system. The mathematical model is derived using coondinates u,v fixed to the rotating shaft (Figure 2). The disc is whirling at angular speed w. The shaft center E is defined in stationary coordinate.

$$r = z + iy \text{ and in rotating coordinates } u \ \& \ v \qquad (1)$$

$$W = u + iv$$

The velocity (v) and acceleration (a) components of the centroid G in rotating coondinates are:

$$V_u = \dot{u} - w\ (v + a_2) \qquad (3)$$

$$V_v = \dot{v} + w\ (u + a_1)$$

and

$$a_u = \ddot{u} - 2w\dot{v} - w^2\ (u + a_1) \qquad (4)$$

$$a_v = \ddot{v} + 2w\dot{u} - w^2\ (v + a_2)$$

Using equations (4) and D'Alembert's principle, the inertia forces along with the restoring forces. Considering K_u and K_v as the shaft stiffness in principal directions the equations of motion are:

$$\begin{aligned} &\ddot{u} - 2w\dot{v} + (w_1^2 - w^2)u - (\epsilon w_n^2 \cos wt)u = a_1 w^2 + g \cos wt \\ &\ddot{v} + 2w\dot{u} + (w_n^2 - w^2)\ v = a_2 w^2 - g \sin wt \end{aligned} \qquad (5)$$

where a_1 and a_2 are eccentricity dimension in the two coordinates u&v and ϵ^2 represents the reduction in rotor stiffness due to crack onset.

Equations (5) are linear with time varying coefficients. They can be reduced to the form of Mathieu equations. Since the time varying term is periodic and single valued, any of the

The straight forward perturbation method is found to be convenient.

The solution generally consists of three parts, free vibration unbalance and gravity excitations. With the equations being linear unbalance and gravity excitations can be accounted for separately and added together. Reference 6 includes a detailed description for those who are interested.

FREE VIBRATION SOLUTION

Seeking a first order uniform expansion of the solution for (1) for small values of 6 the form

$$u = u_o + \epsilon u_1$$

$$v = v_o + \epsilon v_1$$

Substituting by 2 into 1, and equating coefficients of like powers of ϵ one obtains:

$$\begin{bmatrix} \ddot{u}_o \\ \ddot{v}_o \end{bmatrix} + 2w \begin{bmatrix} o & -1 \\ 1 & o \end{bmatrix} \begin{bmatrix} \dot{u}_o \\ \dot{v}_o \end{bmatrix} + \begin{bmatrix} w_1^2 - w^2 & o \\ o & w_n^2 - w^2 \end{bmatrix} \begin{bmatrix} u_o \\ v_o \end{bmatrix} = \begin{bmatrix} o \\ o \end{bmatrix} \quad (3)$$

ii ϵ^1 order

$$\begin{bmatrix} \ddot{u}_1 \\ \ddot{v}_i \end{bmatrix} + 2w \begin{bmatrix} o & -1 \\ 1 & o \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \end{bmatrix} + \begin{bmatrix} w_1^2 - w^2 & o \\ o & w_n^2 - w^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} w_n^2 \cos wt \\ o \end{bmatrix} \begin{bmatrix} u_o \\ v_o \end{bmatrix} \quad (4)$$

The solution of (3) is written on the form

$$u_o = U_o e^{iwot}, \text{ and, } v_o = V_o e^{iwot} \quad (5)$$

where w_o is the frequency of vibration of the system in rotating coordinates and the frequency of vibration in the stationary coordinates ws

$$w_s = w_o + w \quad (6)$$

substituting by (5) in equations (3) the following is obtained.

$$\begin{bmatrix} (w_1^2 - w^2 - w_o^2) & -2\, i w w_o \\ 2 i w w_o & (w_n^2 - w^2 - w_o^2) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (7)$$

when the frequency equation can be obtained:

$$w_o^4 - (w_1^2 + w_n^2 + 2w^2)\, w_o^2 + (w_1^2 - w^2)(w_n^2 - w^2) = 0 \qquad (8)$$

Let $P = \frac{w}{wn}$, $P_1 = \frac{w_1}{w_n}$, $P_s = \frac{w_s}{w_n}$

where $w_s = w_o + w$ is the frequency in stationary axes (x, y).

$$P_s = P \pm (\tfrac{1}{2}(P_1^2 + 1 + 2P^2) \pm (\tfrac{1}{2}(P_1^2 + 1 + 2P^2)^2 - (P_1^2 - P^2)(1 - P^2)^{\frac{1}{2}} \qquad (9)$$

For $\epsilon = 0.4, (P_1^2 = .6)$, the above equation is plotted in Figure (3). There are four real roots for $\frac{w_s}{w_n}$ except when

$w_1 < w < w_n$ at which there are only two real roots. There are also two coincident roots as shown in Figure (3), which also fall on synchronous whirl line.

Substituting by u_o and v_o into equations 4 we get the solution on the form.

$$u = U_1 e^{\,i(wo + w)t} + U_2 e^{\,i(w_o - w)t} \qquad (10)$$

To this order, the solution contains several frequency resonances in addition to the one bounded by the frequencies w_{o1} and w_{o2}. These frequencies are:

$w_{o1} + w$, $w_{o2} + w$, $w_{o1} + w_{o2}$, $w_{o1} + 2w$, ..

$(w_{o1} - w_{o2})$

DYNAMIC RESPONSE OF CRACKED SHAFT

In the following the equations of motion are solved for unbalance and gravity excitations so that effect of crack on rotor response is investigated.

A. EFFECT OF OUT OF BALANCE

Equation (1) was solved with appropriate values of unbalance.

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Equations of motion with unbalance excitation are as follows:

$$\ddot{u} - 2w\dot{v} + (w_1^2 - w^2)\,u - \epsilon w_n^2 \cos(wt) = a_1 w^2 \qquad (11)$$

$$\ddot{v} + 2w\dot{u} + (w_n^2 - w^2)\,v = a_2 w^2$$

Following the same procedure as before the unbalance response to a first order approximation may be written as:

$$x = Z_o \cos(wt + \phi_o) + \epsilon\left[\frac{A+B}{2}\cos(2wt) + \frac{A-B}{2}\right] \qquad (12)$$

$$y = Z_o \sin(wt + \phi_o) + \epsilon\frac{A-B}{2}\sin(2wt)$$; Z_o and ϕ_o are given in Appendix 1

where $A = \dfrac{P^2}{P_1^2 - P^2} \quad \dfrac{1 - 2p^2}{(P_1^2 - 2P^2)(1 - 2P^2) - 4P^4}$,and,

$$B = \frac{2P^4}{(P_1^2 - P^2)\quad (P_1^2 - 2P^2)(1 - 2P^2) - 4P^4}$$

B. EFFECT OF GRAVITY ON A CRACKED SHAFT

The effect of gravity is expected to play an important role for the horizontally mounted cracked shafts as in the case of linear asymetric shafts. The response to gravity excitation is studied by solving equations (5) with gravity forces. As discussed earlier for unbalance the following expressions are obtained for gravity response normalized to static deflection.

$$x_g = \frac{C+D}{2}\cos 2wt + \epsilon\frac{C-D}{2} + \left[(M + q/2)\cos wt + (L + q/2)\cos 3wt\right]\epsilon$$

$$yg = \frac{C+D}{2}\sin 2wt + \left[(M - q/2)\sin wt + (L - q/2)\sin 3wt\right]\epsilon \qquad (13)$$

where $C = \dfrac{1 - 4P^2}{(P_1 - 2P^2)(1 - P^2) - 4P^4}$; $D = \dfrac{-(P_1^2 - 4P^2)}{(P_1 - 2P)(1 - P) - 4P}$

$L = \dfrac{2P^2\,C}{(P_1^2 - 5P^2)(1 - 5P) - 16P^4}$; $N = \dfrac{C(1 - 5P^2)}{2(P_1^2 - 5P^2)(1 - 5P^2) - 16P^4}$

$q = N - L$

The dynamic response of a cracked shaft to both imbalance and gravity forces is obtained by adding equations (12) and (13) together.

DISCUSSION

Reviewing the dynamic behaviour of the cracked shaft as obtained analytically, the following conclusions are drawn.

Regarding the free vibration solution:

1. A tendency to instability at speeds between w_1 and w_n where w_1 and w_n are given by equation (0).

2. Several combination of resonant frequencies exist as a result of the periodic variation of stiffness as a result of crack onset.

Then the solution for the unbalance response shows the following distinct changes

a) Unbalance generator vibration at synchronous frequency as well as components at twice / rev. The synchronous vibration exhibits resonance behavior at frequencies w_1 and w_n. It is noticed that the rotor motion is stable between these two resonances.(Fig,4)

b) The twice/rev amplitude excites several resonantes at frequencies corresponding to

w_1 and $w_1 / (2 + 2 w_1^2)^{\frac{1}{2}}$

Under gravity the cracked shaft attempts to oscillate between the stiff and weakened positions. The motion of the shaft contains one per revolution, twice/rev and 3/rev as indicated by equation (13) and Figures (7-9) illustrated as a function of speed ratio. The effect of gravity excited forces is significant in the low speed range, while unbalance forces are domainant at high speeds.

Several resonant frequencies are shown in the zone bounded by w_1 and w_n. Also some resonant peaks occur at approximately ½, 1/3 which are defined as subcriticals. Shafts motion indicated by Figure 10 and 11 is almost sinusoidal. The shaft orbit is clearly elleptic as given by Figure 12.

EXPERIMENTAL RESULTS AND CONCLUSIONS

The predictions of the theoritical model outlined above have been compared with results from the experimental rig shown in figure 13,14 and described in detail in reference (6) A saw cut 3 mm deep was machined. The majority of the results from the experiment will be published elsewhere but the major conclusion drawn from experiment and confirmed by calculations are

1) The low speed vibrations are dominantly excited by gravity.

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Gravity induced vibrations contain strong second and third harmonics. as indicated by Figure (15).

2) Spectrum at speed 3500 Figure (16) has shown a large synchronous component, while higher harmoics are attenuated. Large synchronous amplitude is an indication that unbalance forces are significant. Unbalance has been found to show synchronous and 2/rev vibrations. The nonlinearity present in the system is due to the large deflection experienced by the shaft.

3) The shaft showed a severe instability at 3800 rpm which is 0.9 of the shaft critical speed. The shaft could be rotated beyond this speed as damage was eminant. This unstable operation is demonstrated by calculations.

4) As a result of the weakening effect of the crack, the rotor deflection is observed to increase appreciably. A subcritical peak is observed at speed 2500 rpm shown by the response curve figure (17). These phenomena have been indicated analytically.

References

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APPENDIX 1

$$Z_o = P^2\left(1/(P_1^2-P^2)^2 + a^2/(1-P^2)^2 \right)^{\frac{1}{2}} \quad ; \quad \phi_o = \tan^{-1} \frac{a(P_1^2-P^2)}{1-P}$$

SECOND A.M.E. CONFERENCE

6 - 8 May 1986 , Cairo

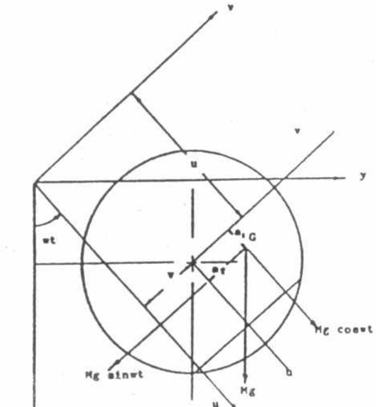
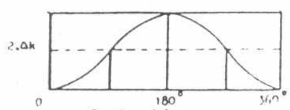


Fig.2 Whirling of a rotor with crack



Crack model
Fig (1)

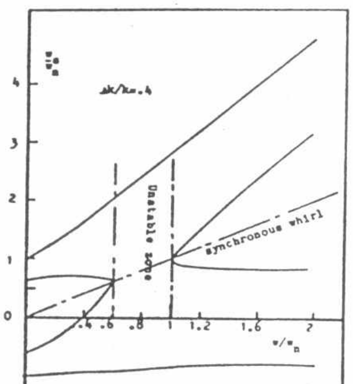


Fig 3 Free whirling of a balanced vertical cracked shaft.

UNBALANCE RESPONSE – FIRST ORDER

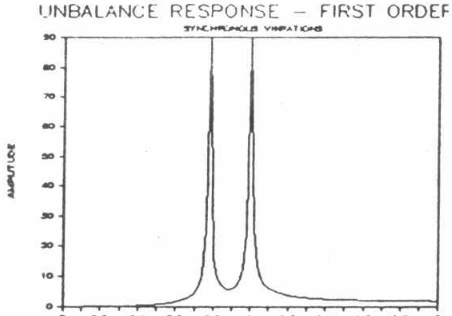


Fig 4

GRAVITY RESPONSE–FIRST ORDER

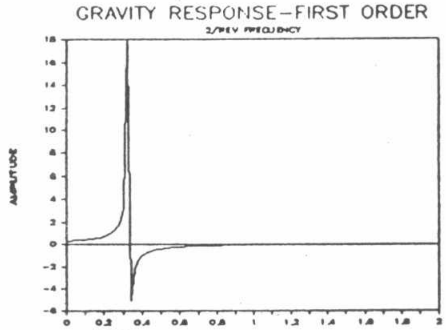


Fig 7

GRAVITY RESPONSE–SECOND ORDER

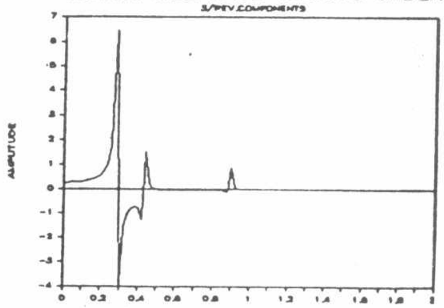


Fig 8

GRAVITY RESPONSE–2ND ORDER

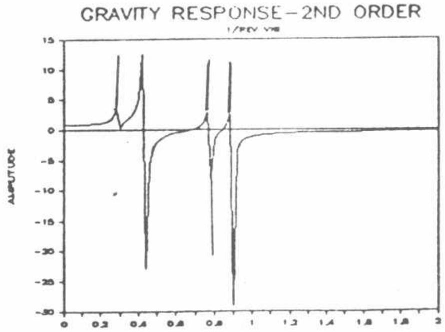


Fig 9

TIME DOMAIN (CRACKED &ASSY. SHAFT)

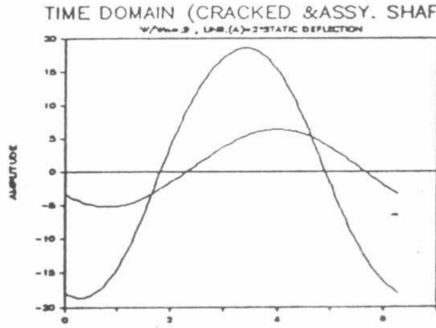


Fig 10

TIME DOMAIN (CRACKED &ASSY. SHAFT)

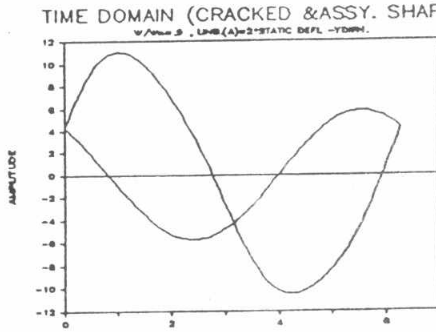


Fig 11

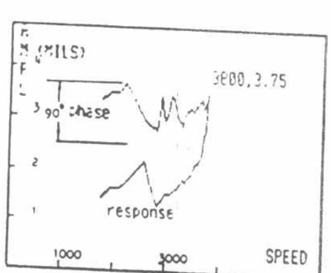


Fig (17)

CRACKED SHAFT ORBIT

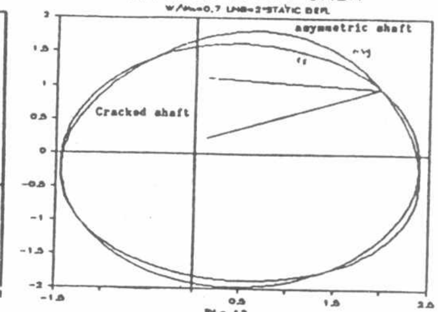


Fig 12

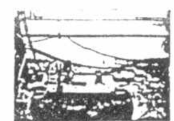


Fig 13

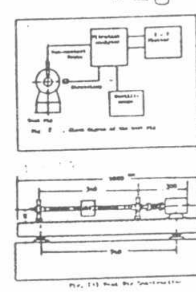
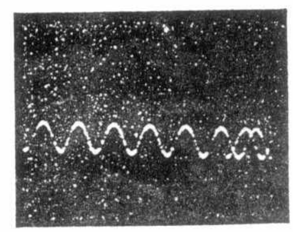


fig (14)



Time history of cracked shaft
Speed= 2500 rpm

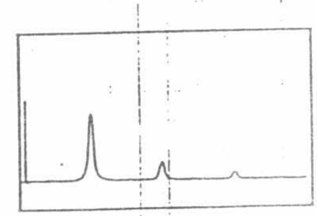


Fig 15 Freq. spectrum Speed=2500 rpm