A RIGIDITY THEOREM FOR SURFACES IN RIEMANNIAN 3-SPACES.

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ABSTRACT

Let $M : D \rightarrow V^3$ and $\bar{M} : D \rightarrow \bar{V}^3$ ($D \subseteq \mathbb{R}^2$) be two isometric surfaces in the Riemannian spaces $V^3$ and $\bar{V}^3$ with curvatures $R$, $\bar{R}$ respectively. We shall prove that the second fundamental forms of the two surfaces are the same provided that:

1. The Gaussian curvature $K$ of $M$ is positive.
2. $M$ and $\bar{M}$ have the same second fundamental form on $\partial D$.
3. For each $d \in D$, $L_d: T(V^3) \rightarrow T(\bar{V}^3)$ is the isometry determined by $M(d)$, $\bar{M}(d)$, $\partial d$ $\in \partial D$, $d \subseteq \partial D$, and $L_d[R(x,y)z] = \bar{R}(L_d x, L_d y) L_d z$ for all tangent vectors $x, y, z \in T(\bar{M})$.

Also it is shown that the two isometric surfaces $M$ and $\bar{M}$ satisfying the above conditions have the same Gaussian and mean curvatures at corresponding points.
INTRODUCTION

It is known that the first fundamental forms I of two isometric surfaces are the same. This is not the case for the second fundamental forms II. However, A. Švec [3] studied the conditions for two infinitesimal surfaces to have the same second fundamental form. He proved that two infinitesimal isometric surfaces in $E^3$ have the same second fundamental form, that is the variation in the second fundamental form $\delta II = 0$ on $M$, provided that the Gaussian curvature $K > 0$ on the surface $M$, and there is a function $\lambda: M \rightarrow \mathbb{R}$ such that the variation of the second fundamental form $\delta II = \lambda I$ on $M$.

Our aim in this paper is to generalize Švec's theorem from the case of infinitesimal isometric surfaces in $E^3$ to the case of the two general isometric surfaces in Riemannian 3-spaces.

THE RIGIDITY THEOREM

Theorem: Let $V^3$, $\bar{V}^3$ be two Riemannian 3-spaces with curvatures $R$, $\bar{R}$ respectively. Let $D \subset \mathbb{R}^2$ be a bounded domain, and let $M: D \rightarrow V^3$, $\bar{M}: D \rightarrow \bar{V}^3$ be two surfaces, such that:

i) $M$ and $\bar{M}$ are isometric.

ii) the Gaussian curvature of $M$ is $K$ and $K > 0$.

iii) For each $d \in D$, let $L_d: T(V^3)_{M(d)}(V^3)$ be the isometry determined by the condition that its restriction to $T(M)$ satisfies $L_dM = M(d)$ and $L_d\{R(x,y)z\} = R(L_dx, L_dy)L_dz$ for each $d \in D$ and all $x, y, z \in T(M)$.

iv) $II$ and $\bar{II}$ are the second fundamental forms of $M$ and $\bar{M}$ respectively, and $II = \bar{II}$ on the boundary $\partial D$.

Then $II = \bar{II}$ on $D$.

Proof: In the Riemannian space $V^3$, let $M: D \rightarrow V^3$ be a surface. For each point $m \in M$ associate an orthonormal frame $\{m, v_i\}$, $i = 1, 2, 3$. Hence there are differential forms $w^i, \omega^j_i$ on $D$ such that

$$dm = \sum_{i=1}^{3} w^i v_i, \quad dv_i = \sum_{j=1}^{3} \omega^j_i v_j, \quad w^i + \omega^j_i = 0 \quad (i, j = 1, 2, 3), \quad (1)$$

with the structure equations...
\[ d\omega^i = \sum_{j=1}^{3} \omega^j \wedge \omega^i_j, \quad d\omega^j_i = \sum_{k=1}^{3} \omega^k \wedge \omega^j_{k} - \frac{1}{2} \sum_{k,L=1}^{3} R^j_{ikL} \omega^k \wedge \omega^L + R^j_{ikL} + R^j_{ikL} = 0 \] (2)

Since \( dm \) lies in the tangent plane \( T_m(M) \), hence from (1) we have
\[ \omega^3 = 0. \] (3)

The exterior differential of (3) gives
\[ \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 = 0, \] (4)

and hence there exist functions \( a, b, c : D \rightarrow \mathbb{R} \) such that
\[ \omega^3 = a \omega^1 + b \omega^2, \quad \omega^3 = b \omega^1 + c \omega^2. \] (5)

The first and second fundamental forms of \( M \) are given respectively by
\[ I = (\omega^1)^2 + (\omega^2)^2, \quad II = \omega^1 \omega^3 + \omega^2 \omega^3 = a(\omega^1)^2 + 2b \omega^1 \omega^2 + c(\omega^2)^2. \] (6)

The Gaussian and mean curvatures of \( M \) are given respectively by
\[ K = ac - b^2, \quad 2H = a + c. \] (7)

Let \( V^3 \) be another Riemannian space, \( \bar{M} : D \rightarrow V^3 \) be another surface. For each point \( \bar{m} \in \bar{M} \) associate an orthonormal frame \( \{ \bar{m}, \bar{v}_i \} \), \( i = 1, 2, 3 \). Hence there are differential forms \( \bar{\omega}^i, \bar{\omega}^j \) on \( D \) such that
\[ d\bar{m} = \sum_{i=1}^{3} \bar{\omega}^i \bar{v}_i, \quad d\bar{v}_i = \sum_{j=1}^{3} \bar{\omega}^j \bar{v}_j, \quad (i, j = 1, 2, 3) \] (8)

\[ d\bar{\omega}^i = \sum_{j=1}^{3} \bar{\omega}^j \wedge \bar{\omega}^i_j, \quad d\bar{\omega}^j_i = \sum_{k=1}^{3} \bar{\omega}^k \wedge \bar{\omega}^j_k - \frac{1}{2} \sum_{k,L=1}^{3} R^j_{ikL} \bar{\omega}^k \wedge \bar{\omega}^L, \]

\[ R^j_{ikL} + R^j_{ikL} = 0, \quad (k, L = 1, 2, 3). \]

Since \( \bar{M} \) is isometric to \( M \), then we can choose the frame \( \{ \bar{m}, \bar{v}_i \} \) in such a way that
\[ \bar{\omega}^1 = \omega^1, \quad \bar{\omega}^2 = \omega^2. \] (9)

Let us write
\[ \bar{\omega}^j_i = \omega^j_i + \gamma^j_i \] (10)

From (9)
\[ \omega \wedge \tau^2 = \frac{1}{2} \omega \wedge \tau^2 = 0, \tag{11} \]

hence
\[ \tau^2_1 = 0 \quad \text{and} \quad \bar{\omega}^2 = \omega^2 \tag{12} \]

Further we get from (2), (3), (8), (9) and (12)
\[ \omega \wedge \tau^3_1 + \omega \wedge \tau^3_2 = 0, \]
\[ \omega^3_1 \wedge \tau^3_2 + \omega^3_1 \wedge \tau^3_2 + \omega^3_1 \wedge \tau^3_2 = (R^2_{112} - \frac{R^2}{R^2_{112}}) \omega \wedge \omega^2, \]
\[ d \tau^3_1 = \omega^2 \wedge \tau^3_2 + (R^3_{112} - \frac{R^3}{R^3_{112}}) \omega \wedge \omega^2, \]
\[ d \tau^3_2 = -\omega^2 \wedge \tau^3_2 + (R^2_{212} - \frac{R^2}{R^2_{212}}) \omega \wedge \omega^2. \tag{13} \]

Equation (13.1) implies that there exist functions \( R_1, R_2, R_3 : D \rightarrow R \) such that:
\[ \tau^3_1 = R_1 \omega + R_2 \omega^2, \quad \tau^3_2 = R_2 \omega + R_3 \omega^2. \tag{14} \]

From (14) the second fundamental form of \( M \) is then
\[ \bar{\Pi} = \Pi + R_1 (\omega^2) + 2R_2 \omega^1 \omega^2 + R_3 (\omega^2)^2. \tag{15} \]

The exterior differentiation of (12) gives
\[ \omega^3_1 \wedge \omega^3_2 + R^2_{112} \omega^1 \wedge \omega^2 = \omega^3_1 \wedge \omega^3_2 + \frac{R^2}{R^2_{112}} \omega^1 \wedge \omega^2, \]

From (5)
\[ (ac - b^2) + R^2_{112} = (ac - b^2) + \frac{R^2}{R^2_{112}}, \]

from (7) it follows that
\[ K + R^2_{112} = \bar{K} + \frac{R^2}{R^2_{112}}. \tag{16} \]

From (5), (14), (13.2) and (16) we get
\[ aR_3 - 2bR_2 + cR_1 + R_1 R_3 - R^2_2 = \bar{K} - K. \tag{17} \]

From (13.3,4) and (14)
\[ (dR_1 - R_2 \omega^2_1) \wedge \omega^1 + \left\{ dR_2 + (R_1 - R_3) \omega^2_1 \right\} \wedge \omega^2 = (R^3_{112} - \frac{R^3}{R^3_{112}}) \omega^1 \wedge \omega^2. \tag{18} \]
and hence there exist functions $S_1, \ldots, S_4 : D \rightarrow \mathbb{R}$ each that
\[
\begin{align*}
\frac{dR_1}{2} (R_1 - R_3) \omega_1^2 + (dR_3 + 2R_2 \omega_1^2) \omega_2^2 &= (R_3^2 - R_2^2) \omega_1^1 \omega_2^2, \\
\frac{d}{2} (R_1 - R_3) \omega_1^2 &= S_1 \omega_1^1 + (S_2 + R_3^3) \omega_2^2, \\
\frac{dR_2}{2} + (R_1 - R_3) \omega_1^2 &= (S_2 + R_1^3) \omega_1^1 + (S_3 + R_1^3) \omega_2^2, \\
\frac{dR_3}{2} + 2R_2 \omega_1^2 &= (S_3 + R_1^3) \omega_1^1 + \omega_2^2.
\end{align*}
\]

From (2) and (5)
\[
\begin{align*}
\left\{ \frac{dR_1}{2} (R_1 - R_3) \omega_1^2 \right\} \wedge \omega_1^1 + \left\{ \frac{dR_3 + 2R_2 \omega_1^2}{2} \right\} \wedge \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\left\{ \frac{dR_2}{2} + (R_1 - R_3) \omega_1^2 \right\} \wedge \omega_1^1 + \left\{ \frac{dR_3 + 2R_2 \omega_1^2}{2} \right\} \wedge \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\left\{ \frac{dR_1}{2} + 2R_2 \omega_1^2 \right\} \wedge \omega_1^1 + \left\{ \frac{dR_3 + 2R_2 \omega_1^2}{2} \right\} \wedge \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\end{align*}
\]

and we may write
\[
\begin{align*}
da - 2b \omega_1^2 &= \alpha \omega_1^1 + (\beta + \frac{1}{2} R_1^3) \omega_2^2, \\
\omega_1^1 + (dR_1 + 2R_2 \omega_1^2) \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\omega_1^1 + (dR_2 + 2R_1 \omega_1^2) \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\omega_1^1 + (dR_3 + 2R_2 \omega_1^2) \omega_2^2 &= - \frac{1}{2} \omega_1^1 \omega_2^2.
\end{align*}
\]

On differentiating (17) and substituting from (19) and (21) the coefficient of $\omega_1^2$ vanishes. Hence the coefficient of each of $\omega_1^1$ and $\omega_2^2$ will be equal to zero, which gives
\[
\begin{align*}
(c + R_3) S_1 - 2(b + R_2) S_2 + (a + R_1) S_3 &= - \frac{1}{2} \omega_1^1 \omega_2^2, \\
\end{align*}
\]

In $D$, let us choose coordinates $(u,v)$ such that
\[
\begin{align*}
\omega_1^1 &= r \, du, \quad \omega_2^2 = s \, dv, \quad r = r(u,v) \neq 0, \quad s = s(u,v) \neq 0
\end{align*}
\]
which implies that
\[ \omega_1^2 = - \frac{\partial}{\partial v} \frac{\partial r}{\partial u} du + r^{-1} \frac{\partial \Delta}{\partial u} dv. \]

From (19), (23) and (24) we get
\[
\left\{ \begin{aligned}
\frac{\partial (R_1 - R_3)}{\partial u} du + \frac{\partial (R_1 - R_3)}{\partial v} dv - 4R_2 \left(- \frac{\partial}{\partial v} \frac{\partial r}{\partial u} du + r^{-1} \frac{\partial \Delta}{\partial u} dv \right) = \\
(S_1 - S_3 - R_{212}) r du + (S_2 - S_4 + R_{212}) \Delta dv,
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
\frac{\partial R_2}{\partial u} du + \frac{\partial R_2}{\partial v} dv + (R_1 - R_3) \left(- \frac{\partial}{\partial v} \frac{\partial r}{\partial u} du + r^{-1} \frac{\partial \Delta}{\partial u} dv \right) = \\
(S_2 + R_{212}) r du + (S_3 + R_{212}) \Delta dv.
\end{aligned} \right.
\]

From (25) it follows that
\[
\begin{aligned}
\Delta S_1 &= \frac{\partial (R_1 - R_3)}{\partial u} \frac{\partial r}{\partial u} + \frac{\partial R_2}{\partial v} \frac{\partial r}{\partial u} + r \frac{\partial \Delta}{\partial u} (R_1 - R_3) + 4 \frac{\partial r}{\partial v} R_2 + r \Delta (R_{212} - R_{212}) \Delta u, \\
\Delta S_2 &= \frac{\partial R_2}{\partial u} - \frac{\partial r}{\partial v} (R_1 - R_3) - r \Delta R_{212}, \\
\Delta S_3 &= r \frac{\partial R_2}{\partial u} + \frac{\partial \Delta}{\partial u} (R_1 - R_3) - r \Delta R_{212}, \\
\Delta S_4 &= r \frac{\partial (R_1 - R_3)}{\partial u} + \frac{\partial R_2}{\partial v} - \frac{\partial r}{\partial v} (R_1 - R_3) + 4 \frac{\partial \Delta}{\partial u} R_2 - r \Delta (R_{212} - R_{212}).
\end{aligned}
\]

Now, let us turn our attention to condition (iii) of our theorem, for \( x, y, z \in T(M) \), let
\[ M(d) \]
\[ x = x_1 v_1 + x_2 v_2, \quad y = y_1 v_1 + y_2 v_2, \quad z = z_1 v_1 + z_2 v_2, \quad (x_3 = y_3 = z_3 = 0). \]

We have
\[
R(x, y, z) = \sum_{\ell=1}^3 R^\ell_{ijk} x^i y^j z^k v_\ell, \quad (i, j, k = 1, 2, 3)
\]
\[
= R_{112}^2 x^1 y^2 z^1 v_2 + R_{112}^3 y^1 x^2 z^1 v_2 - R_{112}^2 x^1 y^2 z^1 v_2 R_{112}^3 y^1 x^2 z^1 v_3 \\
- R_{112}^2 x^1 y^2 z^1 v_1 + R_{212}^3 y^1 x^2 z^1 v_3 R_{112}^3 y^1 x^2 z^1 v_2.
\]
\[
\begin{align*}
R_{112} &= \left( x^2 y - x y^2 \right) \left( R_{112}^2 \left( z^2 y - z y^2 \right) - \left( R_{112}^3 + R_{212}^3 z^2 \right) y_3 \right) \\
&= \left( x^2 y - x y^2 \right) \left( R_{112}^2 \left( z^2 y - z y^2 \right) - \left( R_{112}^3 + R_{212}^3 z^2 \right) y_3 \right)
\end{align*}
\]

Since \( L_d v_i = \bar{v}_i \), then

\[
L_d \left\{ R(x,y)z \right\} = \left( x^2 y - x y^2 \right) \left( R_{112}^2 \left( z^2 y - z y^2 \right) - \left( R_{112}^3 + R_{212}^3 z^2 \right) y_3 \right)
\]

From (28), equation (17) can be written in the forms

\[
\begin{align*}
(2a+2c+R_1+R_3)R_1 &- 2(2b+R_2)R_2 - (2a+R_1)(R_1-R_3) = 0 \\
(2c+R_3)(R_1-R_3) &- 2(2b+R_2)R_2 + (2a+2c+R_1+R_3)R_3 = 0
\end{align*}
\]

From (10) and (14) we get,

\[
\begin{align*}
\bar{\omega}_1 \wedge \bar{\omega}_2 + \omega^1 \wedge \omega^2 + (R_1+R_3)\omega^1 \wedge \omega^2 &= 2H \omega^1 \wedge \omega^2 \\
\omega^3 \wedge \omega^1 \wedge \omega^2 &= 2H \omega^1 \wedge \omega^2
\end{align*}
\]

Hence

\[
2(H+H) = 2a + 2c + R_1 + R_3.
\]

Using (31), equation (17) can be written in the forms

\[
\begin{align*}
(2a+2c+R_1+R_3)R_1 &- 2(2b+R_2)R_2 - (2a+R_1)(R_1-R_3) = 0 \\
(2c+R_3)(R_1-R_3) &- 2(2b+R_2)R_2 + (2a+2c+R_1+R_3)R_3 = 0
\end{align*}
\]

From (29) and (22) we get

\[
\begin{align*}
R_1 &= (H+H)^{-1} \left\{ (2b+R_2)R_2 + \frac{1}{2} (2a+R_1)(R_1-R_3) \right\} \\
R_3 &= (H+H)^{-1} \left\{ (2b+R_2)R_2 - \frac{1}{2} (2c+R_3)(R_1-R_3) \right\}
\end{align*}
\]

From (28), (31) and (22) we get

\[
\begin{align*}
K = K \quad (K > 0)
\end{align*}
\]
\[
\begin{align*}
\Delta(c+R_3) \frac{\partial (R_1-R_3)}{\partial u} - 2 \Delta(b+R_2) \frac{\partial R_2}{\partial u} + r(a+c+R_1+R_3) \frac{\partial R_2}{\partial v} &= \\
\frac{f_1(R_1-R_3)}{\partial v} + \frac{f_2 R_2}{\partial v} \\
- \frac{r(a+R_1)}{\partial v} + \Delta(a+c+R_1+R_3) \frac{\partial R_2}{\partial u} - 2r(b+R_2) \frac{\partial R_2}{\partial v} &= \\
\frac{f_3(R_1-R_3)+f_4 R_2}{\partial v}.
\end{align*}
\]

The quadratic form \( \varphi \) of (36) is equivalent to

\[
\varphi = -(a+c+R_1+R_3) \left\{ r^2(a+R_1)\mu^2 + 2r \Delta(b+R_2)\mu \nu + \Delta^2(c+R_3)\nu^2 \right\}.
\]  

(37)

Let the discriminant of \( \varphi \) be \( -\Delta \), then

\[
\Delta = r^2 \Delta^2 (a+c+R_1+R_3)^2 \left\{ (a+R_1)(c+R_3)-(b+R_2)^2 \right\},
\]

(38)

from (7), (17), (31) and \( (a+c+R_1+R_3) = 2\tilde{H} > 0 \) equ (38) reduces to

\[
\Delta = r^2 \Delta^2 K(a+c+R_1+R_3)^2 > 0.
\]

Hence \( \varphi \) is definite and (36) is elliptic, and from (iv) we get \( R_1-R_3 = R_2 = 0 \) in D. From (35) we get \( R_1 = R_2 = R_3 = 0 \) inside D. Then from (33) we get \( H = \tilde{H} \), and from (15) we get \( \Pi = \Pi \) in D, which proves the theorem.

Q.E.D.
CONCLUSION

We conclude that the theorem of A. Švec [3] can be generalized from two infinitesimal isometric surfaces to the case of two general isometric surfaces in Riemannian 3-spaces. Moreover if $V^3 = V^3 = E^3$ the condition (iii) is automatically satisfied since $R_{ijk} = ar{R}_{ijk} = 0$.

REFERENCES


