MILITARY TECHNICAL COLLEGE

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"ON THE OSCILLATIONS OF A BOUNDED LIQUID WITH A TIME DEPENDENT DISCONTINUOUS BOUNDARY".

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ABSTRACT : The stability of the free surface of an inviscid liquid partially filling a rigid rectangular container with an outlet at its base is investigated. The container executes vertical vibrations in the direction of the gravitational field. The initial-boundary value problem has been formulated and solved. The wave height and pressure have been determined. By specializing the obtained results to special cases, it has been found that they agree favourably with previous investigations.

I. INTRODUCTION.

The importance of "Liquid Slosh" phenomenon in many engineering systems is well recognized and extensive investigations of this phenomenon have been conducted and reported. Much of the literature dealing with this phenomenon up to the Year 1966 has been summarized in Abramson's monograph (1) . For later studies Miles (2) , Hunt et al (3) and Nash et al (4) have studied the lateral slosh problem in cylindrical vessels.

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Further on , for the longitudinal slosh in the same vessels, it has been tackled by Pavlovskii (5) and Khandelwal et al (6).

Despite these interesting studies , little is known about the problem of the dynamic behaviour of a liquid in a container when the container has an outlet at its base and is subjected to velocity fluctuation at the outlet due to a pump pressure. To the writer's knowledge , the only works on this problem are Buhta and Yeh (7) and Henrici et al (8) . The first authors have treated the case of axisymmetric slosh of an inviscid liquid in a stationary circular cylindrical container due to the outlet velocity fluctuations at the base of the container. The second authors have studied the determination of the slosh frequencies of an ideal liquid contained in a half-space with a circular or strip-like aperture.

The present paper gives an answer to the problem of the dynamic behaviour of a perfect liquid with a free surface contained in a rectangular vessel with an outlet at its base. The vessel is vibrating in a vertical direction while at the outlet there exists velocity fluctuations. Mathematically , this problem reduces to an initial-boundary value problem with a time dependent discontinuous boundary condition. Naturally, it is expected to be a generalized problem compared with the previous ones , and the object of the present paper is to provide a step towards filling this gap of knowledge in this particular practical hydraulic systems.

2. FORMULATION.

Consider a finite domain D of a liquid bounded by a rigid rectangular container whose dimensions are $2a$ and $2b$. The still level of the liquid in the container is at a height h from its base. There exists at the base a square outlet whose sides is 2ε and is connected with a pump where it brings velocity disturbance at the outlet.

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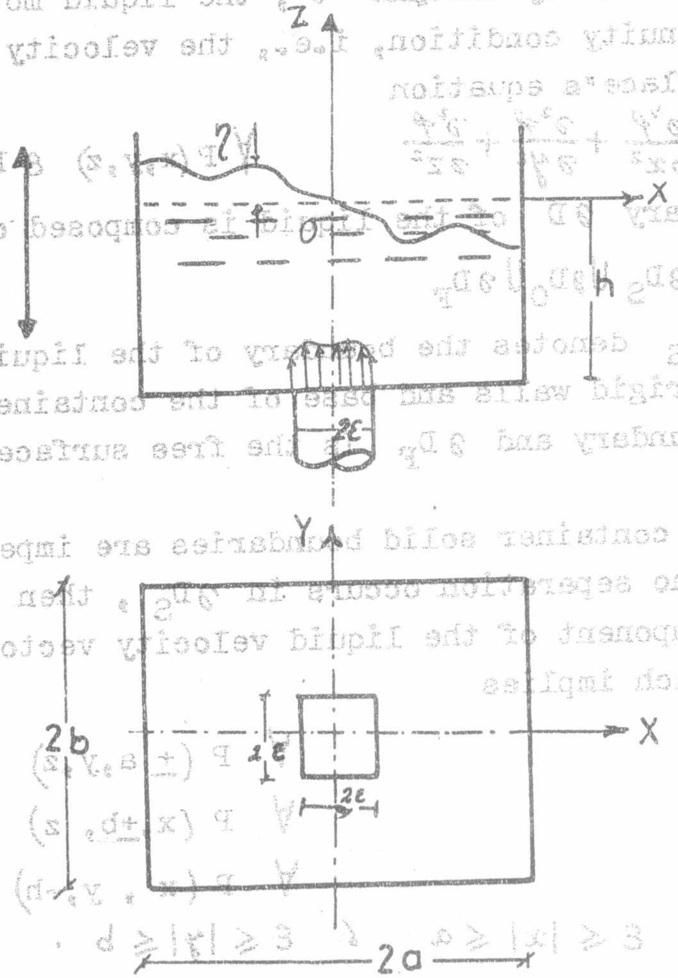

Fig.1. Problem's GEometry.

If the container executes a vertical oscillation defined by $\xi = \xi_0 \cos \omega t$, where ξ_0 is its amplitude and ω is the frequency, then a motion in the liquid is generated. Assuming that the liquid is a perfect one and by adopting a frame of reference R fixed in the container as shown in the annexed figure, one can describe this generated motion of the liquid by following the eulerian representation - for a lagrangian description cf. Ilgomov (9) - . Supposing that the capillary contact effects between the liquid and the container walls are negligible and the motion starts from rest, then there exists a single-valued relative velocity potential function $\varphi(x,y,z;t)$ in which the relative fluid velocity is equal to

$-\operatorname{grad} \phi$. At any instant t, the liquid motion must satisfy the continuity condition, i.e., the velocity potential satisfies LaPlace's equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \forall\ P(x,y,z) \in D \tag{2.1}$$

The boundary ∂D of the liquid is composed of

$$\partial D = \partial D_S \cup \partial D_O \cup \partial D_F \tag{2.2}$$

where ∂D_S denotes the boundary of the liquid in contact with the rigid walls and base of the container, ∂D_O is the outlet boundary and ∂D_F is the free surface boundary.

Since the container solid boundaries are impermeable and if there is no separation occurs in ∂D_S, then the relative normal component of the liquid velocity vector vanishes on ∂D_S, which implies

$$\frac{\partial \phi}{\partial x} = 0 \qquad \forall\ P(\pm a, y, z) \tag{2.3a}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \forall\ P(x, \pm b, z) \tag{2.3b}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \forall\ P(x, y, -h) \tag{2.3c}$$

where $\qquad \varepsilon \leq |x| \leq a \quad , \quad \varepsilon \leq |y| \leq b$.

At the outlet boundary ∂D_O,

$$\frac{\partial \phi}{\partial z} = W(x,y)\, \dot{\zeta}(t) \qquad \forall\ P(x,y,-h) \tag{2.4}$$

where

and $-W(x,y)\, \dot{\zeta}(t)$ is the distribution of the velocity fluctuation at the outlet relative to R.

It is assumed that the amplitude of $\overset{\circ}{\zeta}$ is small such that the amount of liquid flowing in and out of the container can be neglected as compared with the total liquid mass in the container.

For the free surface boundary ∂D_F, the dynamic boundary condition is based entirely on Bernoulli's law. Since the container moves with a central vertical acceleration $-\xi_0 \omega^2 \cos \omega t$, one can consider the liquid motion relative to R as if R is at rest and the gravitational acceleration takes the value $(g - \xi_0 \omega^2 \cos \omega t)$. Thus, by denoting p as the constant pressure at ∂D_F and ρ is the density of the liquid, Bernoulli's law takes the form [cf. Wehausen and Laitone (10)].

$$\frac{1}{2} |\mathrm{grad}\, \varphi|^2 - \frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + (g - \xi_0 \omega^2 \cos \omega t) z = f(t) \qquad (2.5a)$$

where f depends only on the time t and may be put equal to zero.

If the equation of liquid free surface is $z = \eta(x,y;t)$, then an element of liquid on ∂D_F must move so that its velocity component normal to ∂D_F is the same as the normal velocity of ∂D_F itself. Hence, the liquid elevation η must satisfy the condition [cf. Stoker (11), Chap. 2]

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \varphi}{\partial z} \qquad (2.5b)$$

Eq. (2.5b) implies that no spraying or tumbling-over occurs from the liquid.

In order to obtain a tractable mathematical problem, the boundary conditions (2.5) have to be linearized. This can be achieved by expanding all dependent variables in a power series in terms of a fictitious parameter μ about the quiescent state [cf. Nayfeh (12), p.24]. Assuming that these expansions can be differentiated term by term, by virtue of smallness of deviations and their derivatives, the lowest order of Eqs. (2.5) are

$$\eta = \frac{\frac{\partial \varphi}{\partial t}}{g - \xi_0 \omega^2 \cos \omega t} \qquad \forall \; P(x,y,0) \qquad (2.6a)$$

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \varphi}{\partial z} \qquad \forall \; P(x,y,0) \qquad (2.6b)$$

Thus, the posed problem is reduced to an initial-boundary value problem defined by Eqs. (2.1), (2.3), (2.4) & (2.6).

3. SOLUTION.

A solution of Eq. (2.1) that satisfies Eqs. (2.3) is given by

$$\varphi(x,y,z;t) = \dot{p}_0 z + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\frac{m\pi}{a}x \, \cos\frac{n\pi}{b}y \; \times$$
$$\left(\dot{q}_{mn} \cosh r_{mn} z + \dot{p}_{mn} \sinh r_{mn} z\right) \qquad (3.1)$$

Here, q_{mn}, p_{mn} and p_o are generalized coordinates. They are functions of time and the dot denotes as usual time differentiation. The index r_{mn} is

$$r_{mn} = \pi \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{1/2} \qquad (3.2)$$

To satisfy Eq. (2.4), the function $W(x,y)$ is expanded in a double Fourier's series in the form

$$W(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \qquad (3.3)$$

where the spectrum coefficients a_{mn} are evaluated by

$$a_{mn} = \frac{\varepsilon_m \varepsilon_n}{\varepsilon^2} \int_0^{\varepsilon} \int_0^{\varepsilon} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \; W(x,y) \, dx \, dy \qquad (3.4)$$

and

$$\varepsilon_m = \varepsilon_n \begin{array}{ll} = 1 & \text{for } n=m=0 \\ = 2 & \text{for } n=m=1,2,3 \ldots \end{array} \qquad (3.5)$$

Substituting Eq. (3.1) into Eq. (2.4) and making use of Eqs. (3.3 - 5) lead to

$$\dot{p}_o(t) = a_{oo} \dot{\zeta}(t) \qquad (3.6a)$$

and

$$\dot{q}_{mn}(t) \operatorname{Sinh}(r_{mn} h) - \dot{p}_{mn}(t) \operatorname{Cosh}(r_{mn} h) = - \frac{a_{mn}}{r_{mn}} \dot{\zeta}(t) \qquad (3.6b)$$

Integrating Eqs. (3.6 a & b) with respect to time and taking the integration constants to be zero, yield

$$p_o = a_{oo} \zeta(t) \qquad (3.7a)$$

$$p_{mn} = \operatorname{Tanh}(r_{mn} h) \cdot q_{mn} t + \frac{a_{mn}}{r_{mn}} \operatorname{Sech}(r_{mn} h) \zeta(t) \qquad (3.7b)$$

Also, substituting Eq. (3.1) into Eq. (2.6b), integrating, and also putting the integration constant equal to zero give

$$\eta(x,y;t) = -p_o - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r_{mn} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \; p_{mn}(t) \qquad (3.8)$$

Substituting Eqs. (3.1 & 8) into Eq. (2.6a) leads to

$$\ddot{q}_{mn} + r_{mn} (g - \xi_o \omega^2 \cos \omega t) p_{mn} = 0 \qquad (3.9)$$

Eliminating p_{mn} between Eqs. (3.7b and 3.9), brings the differential equation that determines the generalized coordinates q_{mn}, namely,

$$\ddot{q}_{mn} + r_{mn} \operatorname{Tanh}(r_{mn} h)(g - \xi_0 \omega^2 \cos \omega t) q_{mn}$$
$$= - a_{mn} (g - \xi_0 \omega^2 \cos \omega t) \operatorname{Sech}(r_{mn} h) \zeta \qquad (3.10)$$

Considering the special case $m = n = 0$, Eq.(3.10) reduce to

$$\ddot{q}_{00} = - a_{00} (g - \xi_0 \omega^2 \cos \omega t) \zeta = \Theta_0(t) \qquad (3.11)$$

By integrating Eq.(3.11) twice, this gives

$$q_{00}(t) = A + U(t) + [B + V(t)] \, t \qquad (3.12)$$

where A and B are constants of integration to be determined from the initial state of motion. The functions $U(t)$ and $V(t)$ are given by

$$U(t) = - \int t \, \Theta_0(t) \, dt \qquad (3.13a)$$

and

$$V(t) = \int \Theta_0(t) \, dt \qquad (3.13b)$$

For the general case $m, n \neq 0$, by changing to the dimensionless time defined by $\tau = \omega/2 \, t$, Eq.(3.10) can be written in the form

$$q''_{mn} + (\alpha - 2\beta \cos 2\tau) q_{mn} =$$
$$4 a_{mn} \operatorname{Sech}(r_{mn} h) . (\xi_0 \cos 2\tau - \frac{g}{\omega^2}) \zeta \qquad (3.14)$$

where

$$\alpha = \frac{4g}{\omega^2} r_{mn} \operatorname{Tanh}(r_{mn} h) \qquad (3.15a)$$
$$\beta = 2 r_{mn} \xi_0 \operatorname{Tanh}(r_{mn} h) \qquad 3.15b$$

and the prime denotes differentiation with respect to the dimensionless time.

Eq.(3.14) is a nonhomogeneous Mathieu's equation. The complete solution of this equation is constructed by first determining the solutions of the homogeneous part and added to it the particular solution due to the existence of the right hand side term. Solutions for Mathieu functions have been developed and an extensive literature has been concerned with these solutions [cf. Blanch (13)].

Approximate solutions of the homogeneous Mathieu's equation can be constructed in two cases. Firstly, for the parameter β takes a small value, an approximate solution can be obtained by using a Poincaré - type asymptotic expansion [cf. Atherton (14), pp. 13 -17], i.e.,

$$q_{mn}(\tau;\beta) = \sum_{\ell=0}^{\infty} \beta^{\ell}\, q_{mn\ell}(\tau) \qquad (3.16)$$

Substituting Eq.(3.16) into the homogeneous part of Eq.(3.14), expanding and equating the coefficients of identical powers of β lead to

$$q''_{mno} + \alpha\, q_{mno} = 0 \qquad (3.17)$$

The solution of Eq. (3.17) is

$$q_{mno} = M \cos\sqrt{\alpha}\,\tau + N \sin\sqrt{\alpha}\,\tau \qquad (3.18)$$

where M and N are constants.

Also, the explicit first order terms in β suggest that q_{mn1} must satisfy the equation

$$q''_{mn1} + \alpha\, q_{mn1} = 2\beta \cos 2\tau\; q_{mno} \qquad (3.19)$$

Solving for q_{mn1} and combining with q_{mno}, the final solution is

$$q_{mn}(\tau,\beta) = M\; ce^{*}(\tau;\beta) + N\; se^{*}(\tau;\beta) \qquad (3.20)$$

where

$$ce^{*}(\tau;\beta) = \cos\sqrt{\alpha}\,\tau - \frac{\beta}{(2\sqrt{\alpha}+1)}\cos(\sqrt{\alpha}+1)\tau + \frac{\beta}{(2\sqrt{\alpha}-1)}\cos(\sqrt{\alpha}-1)\tau \qquad (3.21a)$$

and

$$se^{*}(\tau;\beta) = \sin\sqrt{\alpha}\,\tau - \frac{\beta}{(2\sqrt{\alpha}+1)}\sin(\sqrt{\alpha}+1)\tau + \frac{\beta}{(2\sqrt{\alpha}-1)}\sin(\sqrt{\alpha}-1)\tau \qquad (3.21b)$$

For the case where β is not small, an approximate solution can be found by reducing to Riccati type [cf. McLachlan (15), §4.81]. Making the substitution

$$q_{mn} = e^{\sqrt{\alpha}\int^{\tau} w\, d\tau} \qquad (3.22)$$

the homogeneous part of Eq.(3.14) becomes

$$\frac{1}{\sqrt{\alpha}}\frac{dw}{d\tau} + w^2 + \rho^2 = 0 \qquad (3.23)$$

where $\rho^2 = \left(1 - \frac{2\beta}{\alpha}\cos 2\tau\right)$ (3.24)

McLachlan (15,p.95) has derived the two solutions of Eq. 3.23 as follows

$$ce^{*}(\tau;\beta) = (\alpha - 2\beta\cos 2\tau)^{-\frac{1}{4}} \cos\left[\sqrt{\alpha+2\beta}\; E_1(\lambda,\tau)\right] \qquad (3.25a)$$

and

$$se^{*}(\tau,\beta) = (\alpha - 2\beta\cos 2\tau)^{-\frac{1}{4}} \sin\left[\sqrt{\alpha+2\beta}\, E_1(\lambda,\tau)\right] \quad (3.25b)$$

The symbol $E_1(\lambda,\tau)$ stands for

$$E_1(\lambda,\tau) = E(\lambda,\tfrac{\pi}{2}) - E(\lambda,\tfrac{\pi}{2}-\tau) \quad (3.26)$$

where $E(\lambda,\tau)$ is an incomplete elliptic integral of the second kind defined by

$$E(\lambda,\tau) = \int_0^{\tau} \sqrt{1-\lambda^2\sin^2\psi}\; d\psi \quad (3.27)$$

Also, $E(\lambda,\frac{\pi}{2})$ is given by

$$E(\lambda,\tfrac{\pi}{2}) = \frac{\pi}{2}\left(1 - \frac{\lambda^2}{4} - \frac{3}{64}\lambda^4 - \frac{5}{256}\lambda^6 - \dots\right) \quad (3.28)$$

and the modulus λ is given by

$$\lambda = 2\left(\frac{\beta}{\alpha+2\beta}\right)^{1/2} \quad (3.29)$$

For finding the particular solution of Eq.(3.14), the Wronskian is clearly a constant term. It is given by

$$W = ce^{*}(0)\, se'^{*}(0) - se^{*}(0)\, ce'^{*}(0) = (\alpha-2\beta)^{-1/2} \quad (3.30)$$

Thus, the particular solution is

$$Sp_1(\tau;\beta) = \sqrt{\alpha-2\beta}\int^{\tau}\left[ce^{*}(T;\beta)\, se^{*}(\tau;\beta) - ce^{*}(\tau;\beta)\, se^{*}(T,\beta)\right]\Theta_1(T)\,dT \quad (3.31)$$

where

$$\Theta_1(\tau) = 4a_{mn}\,\mathrm{Sech}(r_{mn}h)\left(\xi_0\cos 2\tau - \frac{g}{\omega^2}\right)\zeta(\tau) \quad (3.32)$$

Hence, the complete solution of Eq.(3.14) is

$$q_{mn}(\tau) = M\, ce^{*}(\tau;\beta) + N\, se^{*}(\tau;\beta) + Sp(\tau,\beta) \quad (3.33)$$

where M and N are constants of integration to be determined from the initial conditions.

Having obtained the determination of the q_{mn} coordinates, the substitution for q_{mn} from Eq.(3.33) into Eq.(3.7b) leads to the determination of p_{mn} coordinates.

For the determination of the constants A,B,M and N, consideration is given to the initial state of motion of the liquid. Considering that the liquid motion starts so that it has zero initial velocity and zero free surface wave height accompanied by a velocity fluctuation at the outlet of the type $\zeta(t) = \zeta_0 \cos(\omega t+\delta)$, one can get

$$u(x,y,z;0) = \frac{\pi}{a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \times \left[\dot{q}_{mn}(0)\operatorname{Cosh}(r_{mn}z) + \dot{p}_{mn}(0)\operatorname{Sinh}(r_{mn}z)\right] = 0 \quad (3.34)$$

$$v(x,y,z;0) = \frac{\pi}{b}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \times \left[\dot{q}_{mn}(0)\operatorname{Cosh}(r_{mn}z) + \dot{p}_{mn}(0)\operatorname{Sinh}(r_{mn}z)\right] = 0 \quad (3.35)$$

$$w(x,y,z;0) = -\dot{p}_0(0) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn} \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \times \left[\dot{q}_{mn}(0)\operatorname{Sinh}(r_{mn}z) + \dot{p}_{mn}(0)\operatorname{Cosh}(r_{mn}z)\right] = 0 \quad (3.36)$$

$$\eta(x,y;0) = -p_0(0) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn} \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\, p_{mn}(0) = 0. \quad (3.37)$$

Thus, Eqs.(3.34-37) can be satisfied if and only if

$$\dot{q}_{mn}(0) = p_0(0) = \dot{p}_0(0) = p_{mn}(0) = \dot{p}_{mn}(0) = 0 \quad (3.38)$$

Hence, from Eqs.(3.6a & 3.7a), one can get

$$a_{00} = 0 \quad (3.39)$$

Also, from Eq.(3.7b),

$$q_{mn}(0) = -\frac{a_{mn}}{r_{mn}} \operatorname{CoSech}(r_{mn}h)\,\zeta(0) \quad (3.40)$$

Hence, for $m = n = 0$, from Eqs.(3.38, 39 and 40) one can gather that

$$q_{00}(0) = \dot{q}_{00}(0) = 0 \quad (3.41)$$

Substituting Eqs.(3.41) into Eq.(3.12) yields that $A = B = 0$. Also, for $m, n \neq 0$, substituting Eqs.(3.38 & 40) into Eq.(3.33) gives

$$M = -\frac{a_{mn}}{r_{mn}} \operatorname{Cosech}(r_{mn}h)\,\zeta(0) - Sp(0;\beta) \quad (3.42a)$$

$$N = 0 \quad (3.42b)$$

Thus, the final forms for the generalized coordinates q_{mn} and p_{mn} are

$$q_{mn}(\tau) = Sp(\tau;\beta) - \left[\frac{a_{mn}}{r_{mn}} \operatorname{CoSech}(r_{mn}h)\,\zeta(0) + Sp(0;\beta)\right] ce^{*}(\tau;\beta) \quad (3.43)$$

and

$$p_{mn}(\tau) = \operatorname{Tanh}(r_{mn}h)\, Sp(\tau;\beta) - \left[\frac{a_{mn}}{r_{mn}} \operatorname{Sech}(r_{mn}h)\,\zeta(0) + \right.$$

$$+ Tanh(r_{mn}h)\, Sp(o;\beta)\Big]\, ce^{*}(\tau;\beta) + \frac{a_{mn}}{r_{mn}}\, Sech(r_{mn}h)\, \zeta(\tau). \qquad (3.44)$$

4. WAVE CHARACTERISTICS.

From the previous section, it can be easily seen that a pattern of standing waves is occurring on the free surface of the liquid as an outcome of the generated motion. For the determination of this pattern , substituting for p_{mn} coordinates from Eq.(3.44) into Eq.(3.8) give the surface profile of the liquid

$$\eta(x,y;t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\Big[a_{mn}\zeta_0\, Sech(r_{mn}h)\cos\delta + Tanh(r_{mn}h)\, Sp(o;\beta)\Big] \times ce^{*}(\tfrac{\omega}{2}t;\beta) - r_{mn}\, Tanh(r_{mn}h)\, Sp(\tfrac{\omega}{2}t;\beta) - \zeta_0\, a_{mn}\, Sech(r_{mn}h) \cos(\omega t+\delta)\Big\} \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y . \qquad (4.1)$$

For the determination of the pressure distribution in the liquid, Eq.(2.5a) reduce to

$$\frac{p}{\rho} = \frac{\partial\varphi}{\partial t} - (g - \xi_0\,\omega^2\cos\omega t)\, z. \qquad (4.2)$$

Making use of Eqs.(3.1 , 3.7a and 3.39) , $\frac{\partial\varphi}{\partial t}$ is given by

$$\frac{\partial\varphi}{\partial t} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[\ddot{q}_{mn}(t)\, Cosh(r_{mn}z) + \ddot{P}_{mn}\, Sinh(r_{mn}z)\Big] \times \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \qquad (4.3)$$

Substituting for q_{mn} from Eq.(3.43) into Eq.(3.14), one can get

$$\ddot{q}_{mn}(t) = \zeta_0\, a_{mn}\, Sech(r_{mn}h)(\xi_0\,\omega^2\cos\omega t - g)\cos(\omega t+\delta) - r_{mn}\, Tanh(r_{mn}h)(g - \xi_0\,\omega^2\cos\omega t)\Big\{Sp(\tfrac{\omega}{2}t;\beta) - \Big[\frac{a_{mn}}{r_{mn}}\, Cosech(r_{mn}h)\, \zeta_0\cos\delta + Sp(o;\beta)\Big]\, ce^{*}(\tfrac{\omega}{2}t;\beta)\Big\}. \qquad (4.4)$$

Also, from Eq.(3.7b) , one can obtain

$$\ddot{p}_{mn}(t) = Tanh(r_{mn}h).\ddot{q}_{mn}(t) - \frac{a_{mn}}{r_{mn}}\,\omega^2\zeta_0\, Sech(r_{mn}h)\cos(\omega t+\delta). \qquad (4.5)$$

Substituting Eqs.(4.4 & 5) into Eq.(4.3) and further substitution into Eq.(4.2) give the pressure distribution

$$\frac{p}{\rho} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y\Big\{\Big[Cosh(r_{mn}z) + Tanh(r_{mn}h)\, Sinh(r_{mn}z)\Big] \times \Big[\zeta_0\, a_{mn}\, Sech(r_{mn}h)(\xi_0\,\omega^2\cos\omega t - g)\cos(\omega t+\delta) - r_{mn}\, Tanh(r_{mn}h)(g - \xi_0\,\omega^2\cos\omega t)\, Sp(\tfrac{\omega}{2}t;\beta) - \Big(\zeta_0\frac{a_{mn}}{r_{mn}}\, Cosech(r_{mn}h)\cos\delta + Sp(o;\beta)\Big)$$

$$ce^{*}\left(\frac{\omega}{2}t;\beta\right)\Big] - \omega^2 \zeta_0 \frac{a_{mn}}{r_{mn}} Sech\,(r_{mn}h)\cos(\omega t+\delta)$$
$$\left.\sinh(r_{mn}z)\right\} - (g - \xi_0\omega^2\cos\omega t)Z. \qquad (4.6)$$

5. STABILITY

As shown in the previous section , Eq.(4.1) determines the standing waves pattern of the free water surface due to the container vertical vibrating motion accompanied by the flow disturbance at the outlet. What is meant here by stability in this context , is that the wave elevation η form should be always bounded for large values of t , otherwise , it will grow up until it is restrained by nonlinear effects or until the free surface disintegrates. By analyzing Eq.(4.1) , it is readily seen that the boundness of η is dependent on the boundness of the two functions $ce^{*}(t;\beta)$ and $se^{*}(t;\beta)$, since it is obvious that the function $Sp\ (t;\beta)$ is a combination of them. The parameters governing the problem are α and β . The first parameter is mainly defining the frequency of container vibrations while the second is defining the free mode of oscillation multiplied by the amplitude of the container vertical vibrations. In standard works on Mathieu's equation [cf. McLachlan (15)] , the parameter α is determined so as to bring periodic solutions with periodicity of 2π or π in τ . The special values of α are called characteristic numbers. They are tabulated [cf. Blanch (13) , Chap. 20] according to the integral values r=0,1,2.. For even functions [e.g: $ce^{*}(\tau)$] they are labelled s_r and for odd functions [e.g: $se^{*}(\tau)$] they are labelled $\bar{s}_r$. By graphing these characteristics versus β ; a chart is obtained (Ince Chart) which shows the regions of stability boundness and instability of the Functions. Thus, for a general point (α,β) where $\beta \geq 0$ that lies in the region $s_r(\beta)$ and $\bar{s}_{r+1}(\beta)$ the solution is considered stable while if the point (α,β) lies in the region $\bar{s}_{r+1}(\beta)$ and $s_{r+1}(\beta)$ the solution is unstable. Thus, to investigate the stability for a general amplitude ξ_0 and frequency ω the parameters (α,β) are computed from Eq.(3.15) respectively ; for each mode in turn and then locate it on the chart to determine the region on which it lies. Naturally, this is not a rational approach but it is a practical one.

To overcome partially this experimental sorting of testing for stability , a proposal is given here in the present paper which is suitable only for small and large values of the parameter β . For moderate values of β it cannot yet work. Firstly , for small values of β i.e, β^3 and higher quantities can be neglected , the characteristics are given by [cf. Meixner and Schäfke (16)] .

$$
\begin{aligned}
S_0 &= -\tfrac{1}{2}\beta^2 \\
S_1 &= 1+\beta-\tfrac{1}{8}\beta^2 \quad , \quad \bar{S}_1 = 1-\beta-\tfrac{1}{8}\beta^2 \\
S_2 &= 4+\tfrac{5}{12}\beta^2 \quad , \quad \bar{S}_2 = 4-\tfrac{1}{12}\beta^2 \\
S_r &= \bar{S}_r = r^2+\frac{\beta^2}{2(r^2-1)} \qquad r \geqslant 3.
\end{aligned}
\tag{5.1}
$$

Thus , the stability regions are given

$$
\begin{aligned}
-\tfrac{1}{2}\beta^2 &< \alpha < 1+\beta-\tfrac{1}{8}\beta^2 \\
1+\beta-\tfrac{1}{8}\beta^2 &< \alpha < 4-\tfrac{1}{12}\beta^2 \\
4+\tfrac{5}{12}\beta^2 &< \alpha < 9+\frac{\beta^2}{16} \\
r^2+\frac{\beta^2}{2(r^2-1)} &< \alpha < (r+1)^2+\frac{\beta^2}{2r(r+2)} \\
& \qquad r \geqslant 3
\end{aligned}
\tag{5.2}
$$

Secondly , if β is a large quantity which occurs naturally for higher modes of oscillation , the stability is given by the inequality

$$
\alpha < 2^{4r+5}\sqrt{\frac{2}{\pi}}\,\frac{\beta^{\frac{r}{2}+\frac{3}{4}}}{r!}\,e^{-4\sqrt{\beta}} \tag{5.3}
$$

6. SPECIAL CASES.

Case I. $\varepsilon = 0$

The case of $\varepsilon = 0$ means closing the outlet at the base of the container. Thus, Eqs.(3.3) indicate that the coefficients a_{mn} become zero and from Eqs.(3.7) one can obtain

$$p_0(t) = 0 \quad , \quad p_{mn}(t) = q_{mn}(t)\ \mathrm{Tanh}(r_{mn}h) \tag{6.1}$$

Also, Eq.(3.10) reduces to

$$\ddot{q}_{mn} + r_{mn}\,\mathrm{Tanh}(r_{mn}h)\,(g - \xi_0\,\omega^2\cos\omega t)\,q_{mn} = 0 \tag{6.2}$$

The solution of Eq.(6.2) is

$$q_{mn} = M\ ce^*(\tfrac{\omega}{2}t;\beta) + N\ se^*(\tfrac{\omega}{2}t;\beta) \tag{6.3}$$

For determining the constants M and N, the liquid is assumed initially at rest and it possesses a certain initial free surface shape. This last assumption is realistic, for real liquids have always a curvature due to surface tension. From Eqs. (3.34 - 36), one can get

$$\dot{q}_{mn}(0) = \dot{p}_{mn}(0) = 0 \tag{6.4}$$

and from Eq.(3.8)

$$\eta_0(x,y) = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}\, p_{mn}(0)\cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \tag{6.5}$$

where

$$p_{mn}(0) = -\frac{4}{a\,b\,r_{mn}}\int_0^a\!\!\int_0^b \cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\ \eta_0(x,y)\,dx\,dy \tag{6.6}$$

Also, from Eq.(6.1)

$$q_{mn}(0) = -\frac{4\,\mathrm{CoTanh}(r_{mn}h)}{a\,b\,r_{mn}}\int_0^a\!\!\int_0^b \cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\ \eta_0(x,y)\,dx\,dy \tag{6.7}$$

Substituting for $q_{mn}(0)$ and $\dot{q}_{mn}(0)$ from Eqs.(6.7 & 4) into Eq.(6.3) yields

$$M = (\alpha - 2\beta)^{\frac{1}{4}}\ q_{mn}(0) \tag{6.8a}$$

$$N = 0 \tag{6.8b}$$

Hence,

$$q_{mn}(t) = q_{mn}(0)\,(\alpha - 2\beta)^{\frac{1}{4}}\,ce^*(\tfrac{\omega}{2}t;\beta) \tag{6.9}$$

and

$$p_{mn}(t) = q_{mn}(0)\,\mathrm{Tanh}(r_{mn}h)\,(\alpha-2\beta)^{\frac{1}{4}}\,ce^*(\tfrac{\omega}{2}t;\beta) \tag{6.10}$$

Thus, the free surface standing wave pattern is

$$\eta(x,y;t) = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}\,\mathrm{Tanh}(r_{mn}h)\,q_{mn}(0)\,(\alpha-2\beta)^{\frac{1}{4}}\ ce^*(\tfrac{\omega}{2}t;\beta)\cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\,. \tag{6.11}$$

and the liquid pressure distribution is

$$\frac{p}{\rho} = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}\, q_{mn}(0)\, \mathrm{Tanh}(r_{mn}h)\,(\alpha-2\beta)^{\frac{1}{4}}(g-\xi_0\omega^2\cos\omega t)\left[\mathrm{Cosh}(r_{mn}z)+\mathrm{Tanh}(r_{mn}h)\,\mathrm{Sinh}(r_{mn}z)\right]\cos\frac{m\pi}{a}x\,\cos\frac{n\pi}{b}y\; ce^{*}\!\left(\frac{\omega}{2}t,\beta\right) - (g-\xi_0\omega^2\cos\omega t)Z. \quad (6.12)$$

The investigation of the stability of the free surface follows the same route in the general problem.

The results of this section agree exactly with the results of Benjamin and Ursell (17) when neglecting the capillary effects. In this reference , the stability conditions have been discussed on the Ince chart basis.

Case II : $\xi_0 = 0$.

In this particular case, the container is considered stationary and hence the liquid motion is due to flow fluctuations at the outlet of the type $\zeta_0 \cos\omega t$.

Thus, Eq.(3.10) becomes

$$\ddot{q}_{mn} + r_{mn}\, g\, \mathrm{Tanh}(r_{mn}h)\, q_{mn} = -g\zeta_0\, a_{mn}\, \mathrm{Sech}(r_{mn}h)\cos\omega t, \quad (6.13)$$

Writing $\Omega^2 = g\, r_{mn}\, \mathrm{Tanh}(r_{mn}h)$, the solution of Eq.(6.13) is

$$q_{mn} = M\cos\Omega t + N\sin\Omega t - \frac{g\zeta_0}{\omega^2-\Omega^2}\, a_{mn}\, \mathrm{Sech}(r_{mn}h)\cos\omega t. \quad (6.14)$$

and for $\omega \neq \Omega$ the solution (6.14) is stable.

Taking the initial state of zero velocity and zero free surface elevation the constants M and N are given by

$$M = \zeta_0\frac{a_{mn}}{r_{mn}}\left[\frac{g\, r_{mn}}{\omega^2-\Omega^2}\mathrm{Tanh}(r_{mn}h) - 1\right]\mathrm{CoSech}(r_{mn}h). \quad (6.15a)$$

$$N = 0 \quad (6.15b)$$

From Eqs. (3.7 a & b) , one can get

$$p_0(t) = 0 \quad (6.16)$$

and

and

$$p_{mn}(t) = M \operatorname{Tanh}(r_{mn}h)\cos\Omega t + \zeta_0 \frac{a_{mn}}{r_{mn}} \operatorname{Sech}(r_{mn}h) \left[1 - \frac{g\, r_{mn}}{\omega^2 - \Omega^2} \operatorname{Tanh}(r_{mn}h)\right] \cos\omega t. \qquad (6.17)$$

Thus, from Eq.(3.8), the wave pattern reduces to

$$\eta(x,y;t) = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{ M r_{mn} \operatorname{Tanh}(r_{mn}h)\cos\Omega t + \zeta_0 a_{mn} \operatorname{Sech}(r_{mn}h) \left[1 - \frac{g\, r_{mn}}{\omega^2 - \Omega^2}\operatorname{Tanh}(r_{mn}h)\right]\cos\omega t \right\} \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y. \qquad (6.18)$$

Also , substituting Eqs. (6.15 & 17) into Eqs. (4.3 &2) , the pressure distribution is given by

$$\frac{p}{\rho} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \left\{ \left[\frac{g\zeta_0\omega^2}{\omega^2 - \Omega^2} a_{mn} \operatorname{Sech}(r_{mn}h) \cos\omega t - M\Omega^2 \cos\Omega t\right] \left[\operatorname{Cosh}(r_{mn}z) + \operatorname{Tanh}(r_{mn}h)\operatorname{Sinh}(r_{mn}z)\right] - \omega^2 \zeta_0 \frac{a_{mn}}{r_{mn}} \operatorname{Sech}(r_{mn}h)\operatorname{Sinh}(r_{mn}z)\cos\omega t \right\} - gz \qquad (6.19)$$

7. CONCLUSIONS.

The slosh problem of a perfect liquid partially filling a vertically vibrating rectangular container with an outlet at its base is formulated and solved. The stability of the occurring wave pattern has been investigated by a rational proposed procedure. For the case where the container is devoid of outlets, the present analysis is shown to agree with those of Benjamin and Ursell (17) .

The present study is directly applicable to the design of many hydraulic moving and stationary systems subjected to a pump fluctuating pressure. By properly assigning the velocity disturbance function and integrating the pressure on the container walls, the total force exerted on the system due to the liquid motion is known.

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NOMENCLATURE

A,B	=	integration constants
a	=	half-container width
b	=	half-container length
D	=	domain of liquid motion
g	=	acceleration of the gravitational field
h	=	liquid depth
M,N	=	integration constants
m,n	=	mode numbers
p	=	pressure , generalized coordinate
q	=	generalized coordinate
R(x,y,z)	=	container fixed frame of reference
r	=	wave number
t	=	time
u,v,w	=	liquid velocity components
W	=	spatial velocity fluctuation function
α, β	=	parameters
ε	=	half-outlet side
ζ_0	=	amplitude of velocity fluctuation
η	=	wave elevation
ξ_0	=	amplitude of container vibrations
ρ	=	liquid density
τ	=	dimensionless time
φ	=	liquid velocity potential
ω	=	frequency of container vibrations