MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

A STUDY OF THE EQUATION OF THE IMPLIED CURVE FOR ISOPARAMETRIC QUADRATIC TRIANGULAR FINITE ELEMENT

BY

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ABSTRACT

Many practical problems solved by the finite element method have curved boundaries which are mainly approximated by use of isoparametric elements. These curved elements have a wide applications in mechanical engineering, e.g. design of gearing, dynamics of shell structure, ... etc.

The use of isoparametric quadratic triangular elements of Lagrange and Hermite type is well established in the finite element method.

In this paper the derivation of the equation of the implied curve when a curved edge is approximated using isoparametric quadratic triangles. The implied curve depends only on the parameters of the nodes associated with the curved side and does not depend on the basis function used. The general six point isoparametric transformation is analyeed with respect to a triangle with two sides and one curve side. Special case of the transformation is considered which lead to implied curve of the form of symmetric parabola, thus enabling isoparametric transformations to match a variety of boundary shapes.

The above study of the isoparametric quadratic curve is applied on the circle which is divided into equal angle sectors. The continuity of this curve between elements was studied for different angle sectors.

1. INTRODUCTION

In the finite element method applied to the problems involving an enclosed region in two demensions, the triangular elements with straight sides are used. These elements are satisfactory if the original domain has a polygonal boundary. If some part of the boundary is curved, elements with at least one curved side are required if a good approximation of the boundary is needed.

Usually the isoparametric transformation is used to deal with curved boundaries in the finite element method. Isoparametric elements were introduced by Jordan [1] , and Zienkiewicz [2]. Reference to isoparametric elements

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where the quadratic basis functions are defined [4] ,

$$U_1 = p(2p - 1)$$
$$U_2 = q(2q - 1)$$
$$U_3 = (1 - p - q)(1 - 2p - 2q) = r(2r - 1)$$
$$U_4 = 4pq$$
$$U_5 = 4q(1 - p - q) = 4qr$$
$$U_6 = 4p(1 - p - q) = 4pr$$

where $r = 1 - p - q$

U_1, U_2, U_3 are the basis functions at vertices, and

U_4, U_5, U_6 are the basis functions at mid-points.

Here, we consider a special case of a triangle with two straight sides and one curved side as in Fig. 1a. For this case, the transformation equation (2.1), reduces to

$$l = p + \alpha pq \qquad (2.2)$$
$$m = q + \beta pq$$

where

$$\alpha = 4(l_4 - \tfrac{1}{2}) \qquad (2.2a)$$
$$\beta = 4(m_4 - \tfrac{1}{2}) \qquad (2.2b)$$

The quadratic isoparametric transformation between a standard unit triangle and a general curved triangle, of which equation (2.1) is a special case, is given [4] by

$$x = x_3 + (x_1 - x_3)p + (x_2 - x_3)q + Apq$$
$$y = y_3 + (y_1 - y_3)p + (y_2 - y_3)q + Bpq \qquad (2.3)$$

where

$$A = 4(x_4 - (x_1+x_2)/2)$$
$$B = 4(y_4 - (y_1+y_2)/2)$$

As explained in [2] , [3] , and [4], we need the condition that the jacobian is of one sign over an element. In this case we need that $J \geq 0$. It is easy to show the jacobian of the standard quadratic isoparametric transformation of equation (2.2) is

$$J(p,q) = 1 + \beta p + \alpha q \qquad (2.4)$$

The jacobian is linear and therefore will be positive over the triangle, if it is positive at the vertices which it is clear from equation (2.4). We can easily see from this condition the equations (1.2a) and (1.2b) give the conditions

$$x_4 \geq 1/4$$
$$y_4 \geq 1/4$$

as in Fig. 2,

$$V(\alpha_1) = (\alpha_1/2 - H)/\tfrac{1}{2}\alpha_1 \qquad (3.1)$$

where the area of unit circle sector of angle α_1 is $\frac{1}{2}\alpha_1$ and H is the area of the curved isoparametric triangle which is defined by the equation,

$$H = \iint_{\text{curved triangle}} 1 \, dx\, dy = \int_{q=0}^{q=1} \int_{p=0}^{p=1-q} J \, dp\, dq \qquad (3.2)$$

From equation (2.3), the jacobian of this transformation is

$$J = C + \left[B(x_1 - x_3) - A(y_1 - y_3)\right]p + \left[A(y_2 - y_3) - B(x_2 - x_3)\right]q \qquad (3.3)$$

where A and B are defined in equation (2.3)

$$C = (x_1 - x_3)(y_2 - y_3) - (y_1 - y_3)(x_2 - x_3)$$

For the case shown in Fig. 3, the points 2 and 4 are ($\cos\alpha_1$, $\sin\alpha_1$) and ($\cos\theta_4$, $\sin\theta_4$) respectively. Then,

$$H = A/6(+y_2) + B/6(1 - x_2) + C/2 \qquad (3.4)$$

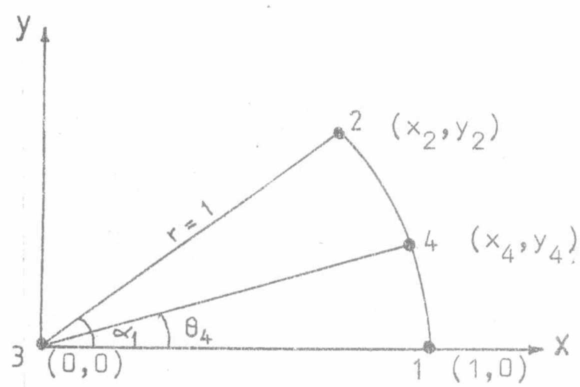

Fig. 3

The condition which gives the minimum value of $V(\alpha_1)$, ($\partial V/\partial\theta_4 = 0$) is

$$\sin\alpha_1 = (1 - \cos\alpha_1)\cos\theta_4 / (1 - \cos^2\theta_4)^{\frac{1}{2}} \qquad (3.5)$$

The manipulation of this relation gives that, $\theta_4 = \alpha_1/2$.

The permissible range (θ_{min}, θ_{max}) for point 4 depends on the sector angle α_1 in Appendix. This makes comparison between sectors difficult for different mid-side node positions. The factor K is choosen as a ratio to define the position of point 4 by the formulae

$$K = (\theta_4 - \delta)/(\alpha_1 - 2\delta)$$

where $0 \angle K \angle 1$

$\delta = \theta_{min}$.

Fig. 4 shows that the $V(\alpha_1)$ and different values of K for different sector angle α_1, the minimum area between the circle and the parabola at K=0.5

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where $\bar{\alpha} = 2(2l_6 - 1)$ and $\bar{\beta} = 2(2m_6 - 1)$

To get the condition of continuity of slope between two adjacent elements we equate equation (4.4) and equation (4.5) to obtain

$$\frac{(y_1 - y_2) + (\alpha y_1 + \beta y_2)}{(x_1 - x_2) + (\alpha x_1 + \beta x_2)} = \frac{(y_2 - y_5) - (\bar{\alpha} y_2 + \bar{\beta} y_5)}{(x_2 - x_5) - (\bar{\alpha} x_2 + \bar{\beta} x_5)} \qquad (4.6)$$

In the present study of continuity of slope of the parabola around the unit circle with mid-side nodes are on the circle, we illustrate the problem with 4 elements as shown in Fig. 7

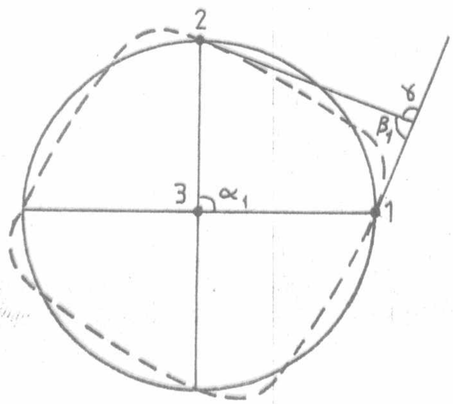

Fig. 7

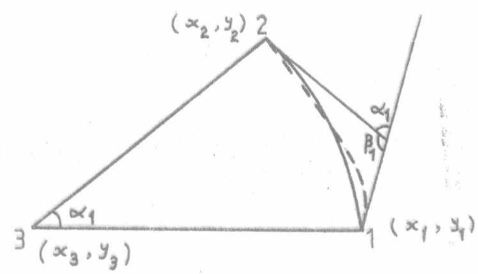

Fig. 8

Apply equation (4.6) between the adjacent element arcs, using polar coordinates for any point i (l_i=cos θ_i, m_i=sin θ_i), we obtain the set of four equations is non-linear and very difficult to solve.

We try a geometric approach to the solution of this set of four equations with the mid-side node on circle. We suppose at each node, the tangents for isoparametric parabola of the two adjacent elements have the same slope to get the continuity in Fig. 7, we have

$$\alpha_1 = 2\pi/n, \qquad \beta_1 = (n - 1). 2\pi/n$$

where α_1 is the sector angle, n number of elements, β_1 the angle between the two tangents at the nodes 1 and 2.
But we have, $\gamma + \beta_1 = 2\pi$ and $\alpha_1 + \beta_1 = 2\pi$
Then,

$$\gamma = \alpha_1$$

Hence, we have the angle of the sector is the same as the angle between the two tangents of the nodes as in Fig. 8, if δ_1 is the slope of the tangent at point (x_1,y_1) from equation (4.3) and δ_2 is the slope of the tangent at point (x_2,y_2) from equation (4.4), then we have

$$\tan \alpha_1 = \frac{\delta_1 - \delta_2}{1 + \delta_1 . \delta_2} \qquad (4.7)$$

5. CONCLUSION

A study for the isoparametric quadratic triangle has been presented. In the first stage, we have shown the derivation of the equation of implied curve which is a parabola. Then, a measure of approximation to the circle using the isoparametric quadratic curve and the C^1 continuity of this curve between elements were studied.

Two main results have been obtained. First, a minimum measure for all approximations studied when the mid-side node is in the middle of the arc. Second, C^1 continuity with the mid-side node on the circle, depends on the sector angle. There is a limit value for the sector angle, smaller than it, the mid-side node lies in the first forbidden area as in Table 1.

In further stage, the study of the C^1 continuity of the parabola between elements when the mid-side node not on the original curve, the circle, will be represented.

6. REFERENCES

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7. APPENDIX: FORBIDDEN ZONES FOR THE MID-SIDE NODE

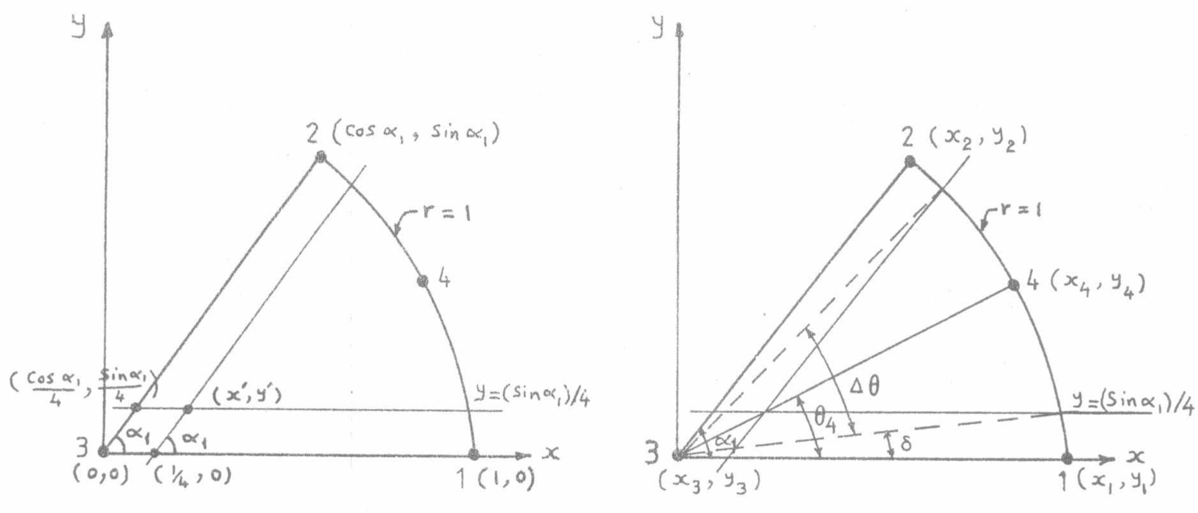

Fig. 9

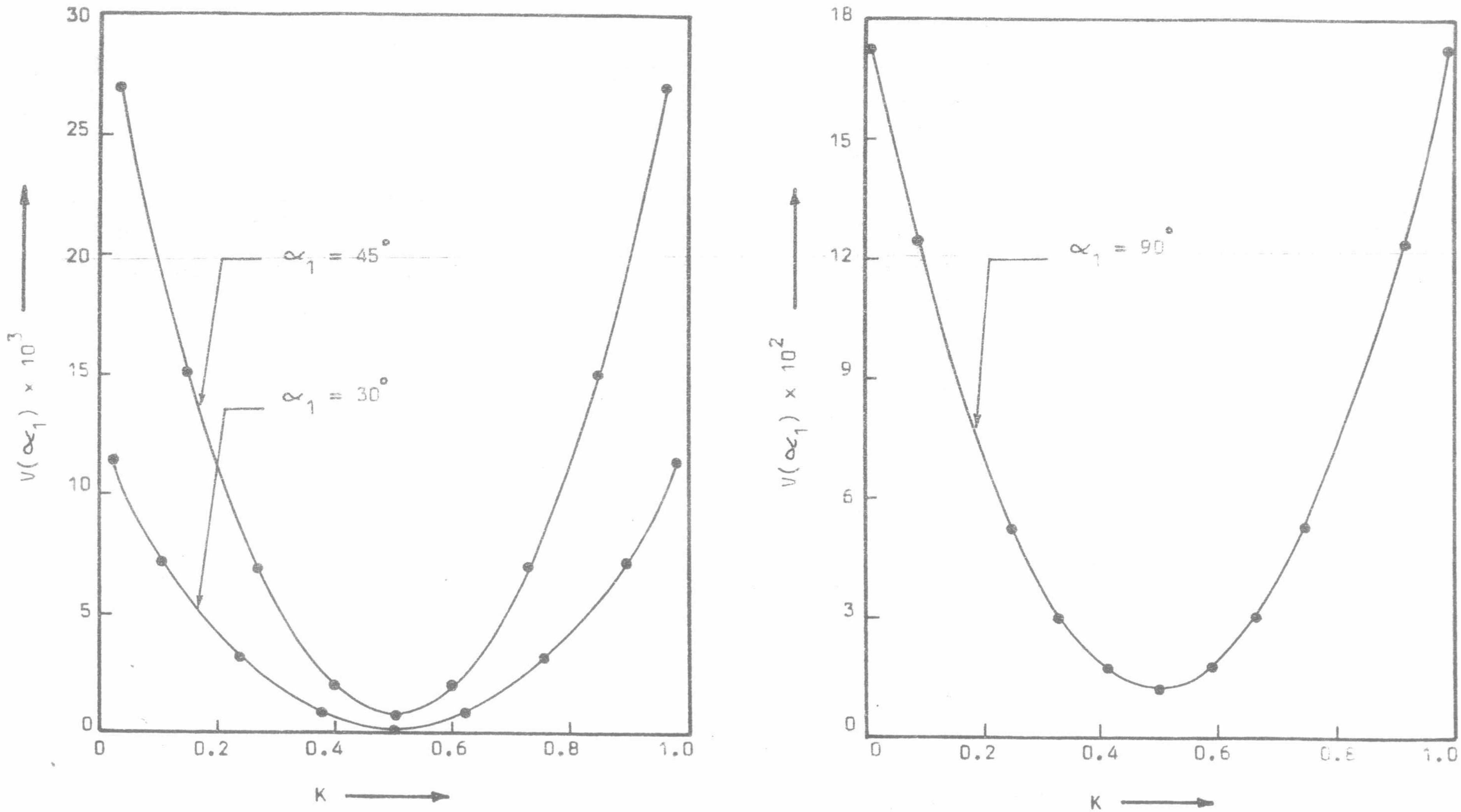
30
25
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1.0
$V(\alpha_1) \times 10^3$
$\alpha_1 = 45°$
$\alpha_1 = 30°$
K
18
15
12
9
6
3
0
$V(\alpha_1) \times 10^2$
$\alpha_1 = 90°$

Fig. 4