A REGULAR BOUNDARY ELEMENT METHOD FOR TWO DIMENSIONAL STRESS ANALYSIS

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ABSTRACT

The Boundary Method is now well established as a valid numerical technique for the solution of field problems, equal to the Finite Element Method in generality and surpassing it in computational efficiency in some cases [1] In this paper is presented a Regular Boundary Element Method as applied to two dimensional stress analysis. It involves the formation of a system of regular integral equations obtained by moving the singularity outside the domain of the given problem [2]. It is shown that continuous elements may be used here after the manner of Finite Element Method [3].

INTRODUCTION

The manifest success of the finite element method which is one of the domain methods has led to progressively increased demands being made of it. In particular, there is increasing pressure to use sophisticated three dimensional geometric models. But, the increased computing overhead in going from two to three dimensional is considerable so that there is some urgency in exploring methods which may be more efficient than the Finite Element Method in three dimensions. Being a Domain Method, with freedoms distributed over the domain of the problem, the Finite Element Method would appear to carry a heavy penalty when compared with a Boundary Technique such as the Boundary Element Method, with freedoms distributed over the boundary only [1].

Central to the method is the generation of Boundary Integral Equations which properly state the problem to be solved in terms of unknown field functions on the boundary only. These equations are usually obtained using the Fundamental solution of the given problem with the singular point located on the boundary [4]. (The equations for the interior solution are obtained similarly , by locating the singular point within the domain of the problem). There ensues an infinite system of singular surface integral equations, one for each boundary point (being generated by moving the singularity around the boundary). The system is discretized by defining boundary elements, after the manner of finite elements, and the resulting finite system of singular integrals are evaluated, thereby giving a system of algebraic equations.

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There are two drawbacks in the singular method (the singular point of the fundamental function is taken on the boundary of the problem) as normally used [2]. Firstly, not only does the accurate evaluation of these singular integrals require careful and special treatment in the neighbourhood of the singular point, but it may also contribute to relatively higher comutational cost. Secondly, the class of problems for which the method well defined, may be unduly restrictive because of divergence of the integrals.

In this paper it is shown that 'Regular Boundary Integral Equations' can quite readily be derived which also properly state the given problem. These are obtained by the simple device of moving the singularity of the fundamental solution outside the domain of the problem. The resulting system of equations tolerates higher order singularities in the solution than previously and requires no special attention to a singular integrand.

The practicality of the method is demonstrated in two dimensional elastostatics. A critical comparison is made of the results obtained using the new approach, the conventional approach and the finite element Method, for quadratic elements.

THEORY

The governing equation for elastostatics in terms of stress field and in the absence of body forces can be written as:

$$\sigma_{ij,j}(u) = 0 \qquad i,j \in \{1,2\} \tag{1}$$

where σ_{ij} are the stress field components for Ω.

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

$u = u(x)$ ... displacement vector (u_1, u_2)

Ω ... domain

x ... coordinate system x_j $\quad j \in \{1,2\}$

Equilibrium on the boundary require the satisfaction of the following boundary conditions:

$$\sigma_{ij} n_j = t_i \qquad i \text{ and } j \in \{1,2\} \tag{2}$$

where n_j are the direction cosines of the normal with respect to x_1, x_2 and t_i are tractions (surface force intensities).

The stresses and strains are related by the constitutive relation for an isotropic body as:

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \qquad i \text{ and } j \ 1,2 \tag{3}$$

where δ_{ij} is the Kronecker delta

λ, μ, are lame's constants which can also be expressed in terms of the modulus of elasticity (Young's modulus) E and Poisson's ratio ν by ;

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \mu = \frac{E}{2(1+\nu)} \tag{4}$$

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Sometimes, μ is written as G which is the shear modulus.

Substitute equation (3) into (1) we can obtain the governing equations of elastostatics in terms of displacements (Navier's equation):

$$\Delta^* u(x) = 0 \tag{5}$$

The symbol Δ^* is a differential operator :

$$\Delta^* u = \lambda \nabla (\nabla . u) + \mu \nabla . (\nabla u) + \mu \nabla . (\nabla u)^T$$

where ∇ is the gradient operator

$\nabla .$ is the divergence operator

$( )^T$ means transposition.

Now we can formulate the elastostatic problem as the following:

Let us consider a vector field function u, defined over a domain Ω and on its boundary Γ , which satisfies Navier's equation within the domain (Figure 1);

$$\Delta^* u = 0 \tag{6}$$

and boundary conditions

$$u_i = \overline{u_i} \quad \text{on } \Gamma_1$$
$$t_i = \overline{t_i} \quad \text{on } \Gamma_2 \tag{7}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $\overline{u_i}$ and $\overline{t_i}$ are given functions

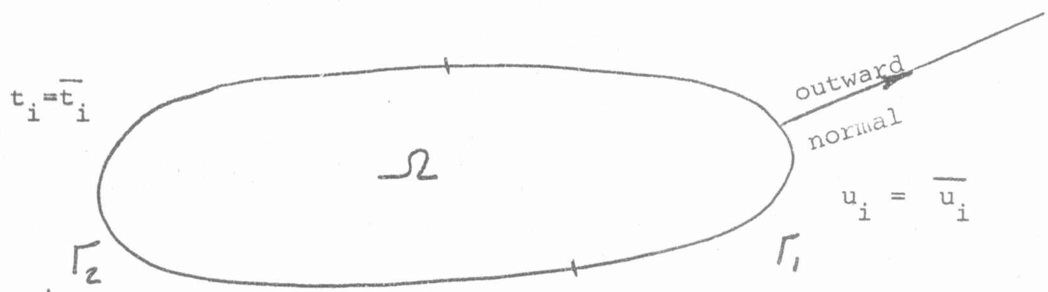


Fig. 1. Notation of elastostatic problem

The weighted residual statement for elastostatic problems can be written as [1] :

$$\int_{\Omega} (\Delta^* u) U^*_{ij} \, d\Omega = \int_{\Gamma_2} (t - \bar{t}) U^*_{ij} \, d\Gamma - \int_{\Gamma_1} (u - \bar{u}) \, T^*_{ij} \, d\Gamma \tag{8}$$

The inverse equation of (8) can be obtained [1] as the following :

$$\int_{\Omega} (\Delta^* U^*_{ij}) u_j \, d\Omega + \int_{\Gamma} U^*_{ij} \, t_j \, d\Gamma = \int_{\Gamma} T^*_{ij} \, u_j \, d\Gamma \tag{9}$$

where $T^*_{ij} = \tau^{(n)} (U^*_{ij})$

$$\tau^{(n)} U = \lambda n (\nabla . U) + \mu n . (\nabla u) + \mu n . (\nabla u)^T$$

U^*_{ij} , T^*_{ij} are weighting functions. The problem is to find a solution such that:

$$\Delta^* U^*_{ij} \equiv 0 \qquad (10)$$

In this way the first integral in equation (9) disappears which reduces the problem to a boundary problem. We need to find the solution to this homogeneous equation (10) (Navier's equation) in order to apply (9) without having to integrate the first term over the domain , which would produce internal unknowns. Away of applying (9) is to use the weighting functions U* as the fundamental (kelvin) solution for the elasticity problem. This type of solution will produce, for each direction, the following equation:

$$C_{ij}(x)\ u_i\ (x) + \int_\Gamma T^*_{ij}(x,y)\ u_j(y)\ d\Gamma(y) = \int_\Gamma U^*_{ij}(x,y)\ t_j(y)\ d\Gamma(y) \quad \text{for all } y \in \Gamma \qquad (11)$$

where $C_{ij}(x)$ coefficient is due to singularities existing in the left hand side integrals of equation (9) .

Equation (11) which is Somigliana's identity can be specialised for an internal ,external and a boundary point and it is the starting point for the singular boundary Element Method in Elastostatics (where the singular point of the fundamental function is taken on the boundary of the problem)

where $C_{ij} = \delta_{ij}$ if x is inside Ω (δ_{ij} =1 if i=j ; 0 i≠j)

$= \frac{1}{2}\delta_{ij}$ if x is on smooth Γ (singular method)

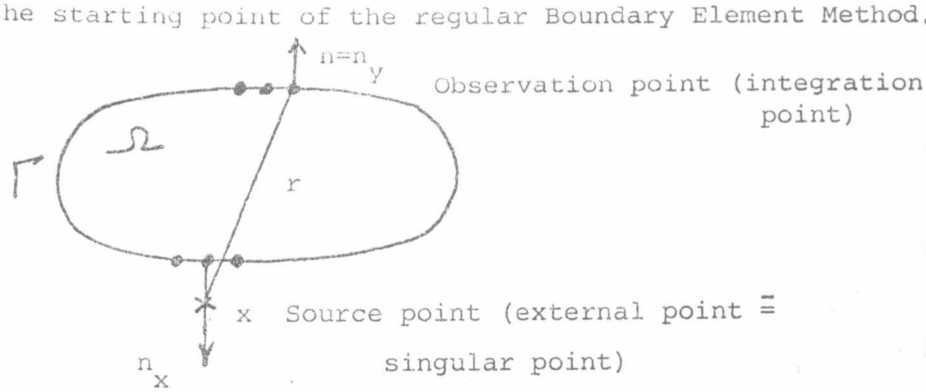
$= 0$ if x is outside Ω (regular method)

If the singular point 'x' is taken outside the domain of the problem (figure 2 a,b), the integration by parts may again be taken with a similar result to equation (11), except that the integrals are regular and, in consequence, C vanishes and equation (11) become:

$$\int_\Gamma T^*_{ij}(x,y)\ u_j(y)\ d\Gamma(y) = \int_\Gamma U^*_{ij}(x,y)\ t_j(y)\ d\Gamma(y) \qquad (12)$$

for $x,y \in \Gamma$

Equation (12) is the starting point of the regular Boundary Element Method.



(a)

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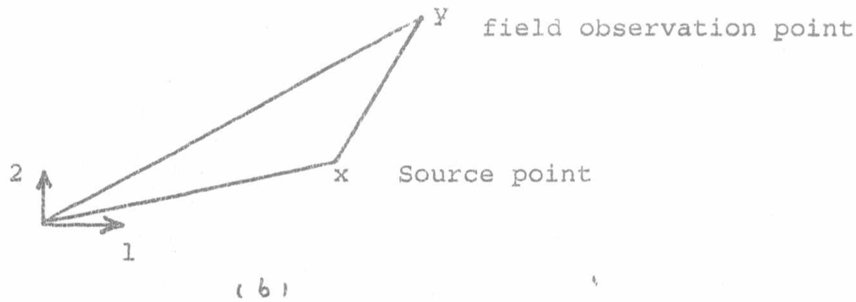

Fig. 2. singular point location

The fundamental (kelvin) solution which satisfies the homogenous equation of elasticity theory (Equation 1) is :

$$U^*_{ij}(x,y) = \frac{1+\nu}{8\pi E(1-\nu)r}\left[(3-4\nu)\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{r^2}\right] \quad (13)$$

where r is the distance between x and y

The traction components corresponding to the kelvin solution are:

$$T^*_{ij}(x,y) = \tau^{(n_y)}\left[U^*_{ij}\right] = \frac{1}{8\pi(1-\nu)r^2}\left\{(1-2\nu)\left[n_i(y)\frac{(x_j-y_j)}{r} - n_j(y)\frac{(x_i-y_i)}{r}\right] + \left[(1-2\nu)\delta_{ij} + 3\frac{(x_i-y_i)(x_j-y_j)}{r^2}\right]n_s(y)\frac{x_s-y_s}{r}\right\}$$

$$i,j \text{ and } s \in \{1,2\} \quad (14)$$

Let us consider the interior problem for the 2D elastostatic problem, and suppose u and t are known for all $x \in \Gamma$. Then formula (11) gives u for all $x \in \Omega$, that is

$$u_i(x) = \int_\Gamma U^*_{ij}(x,y)t_j(y)\,d\Gamma(y) - \int_\Gamma T^*_{ij}(x,y)\,u_j(y)\,d\Gamma(y) \quad (15)$$

$$\text{for all } x \in \Omega$$

According to Hooke's law, the stress tensor σ_{ij} is given in terms of displacement, equation (3). Using u_i, the expression (15) we can obtained σ as:

$$\sigma_{ij}(x) = \int_\Gamma D_{ijk}(x,y)\,t_k(y)d\Gamma(y) - \int_\Gamma S_{ijk}(x,y)\,u_k(y)\,d\Gamma(y) \quad (16)$$

where D and S are functions of material constants $\acute{\nu}$ and $\acute{E}$:

(i) $\grave{\nu} = \nu$ and $\acute{E} = E$ for plane strain

(ii) $\grave{\nu} = \frac{\nu}{1+\nu}$ and $\acute{E} = E(1-\acute{\nu}^2)$ for plane stress.

LOCATION OF SINGULAR POINT:

As stated earlier, in the Regular Boundary Element method the singular point of the fundamental solution kernel function is taken outside the domain of the problem, in contrast with the conventional (singular) method where it is taken on the boundary. In exact arithmetic if there are q freedom nodes in a model, in order to obtain a determinate algebra it is merely necessary to derive q linearly independent kernel functions. This can be achieved by choosing arbitrarily q distinct locations outside the domain of the problem, at which to locate the singularity. In finite arithemtic, it is necessary to locate the singular points reasonably close to the domain in order to avoid ill conditioning problems; on the other hand if the singular points are brought unduly close to the surface a high integration order must be taken near the singularity in order to maintain accurate integration. Clearly, this implies increased computational cost. In order to obtain a systematic approach to the assignment of singular point location it was decided that it should be located on the outward normal from a freedom location, figure 2a, thereby guaranteeing q linearly independent kernel functions. Asystematic study was then made in order to determine the 'best' location of the singular point. This involves a compromise between the need for nicely well conditioned algebraic equations and the need for moderate computer effort. It was found that optimal results were obtained if for each element the singularity was located at a distance from the element, along the outward normal, equal to the minimum distance between freedom nodes for that element.

Once the algebraic equations (12) have been solved, and thereby the values of u and t determined over the boundary surface, the interior solution is determined as usual [3] using equations (15) and (16).

APPLICATIONS

Two 2- dimensional elastostatic problems are analysed using the finite element method,the singular and the Regular Boundary Element Method for quadratic elements and a critical comparison of the results is made.

Thick Walled Cylinder Under Internal Pressure ;

The exact solution of these plane strain problem is well known and given by Lame's solution [5]. The cylinder with inner radius r_1 and outer radius r_2 is considered to have free ends and subjected to uniform internal pressure p, with no pressure applied on the outer surface. Only a quarter of the cylinder is considered for analysis due to symmetry. The boundary conditions are specified in a way to avoid rigid body motion i.e. along AB zero displacement is prescribed in X direction and CD is fixed in y direction, as shown in (Figure 3a,b). The numerical values for the problem are assumed to be :

$$r_1 = 10 \text{ mm} \quad , \quad r_2 = 20 \text{ mm} \quad , \quad p = 2.0 \text{ daN/mm}^2$$

The modulus of Elasticity is taken as 21000 daN/mm^2 and Poisson ratio as 0.3.

The boundary of the domain was divided into 12 quadratic curved elements in a boundary element discretization. The domain was divided into 9 quadratic elements in a finite element discretization, (Figure 3a,b). Computed stresses and displacements at the surface and interior of the domain are shown in figures (4,a,b) together with known values.

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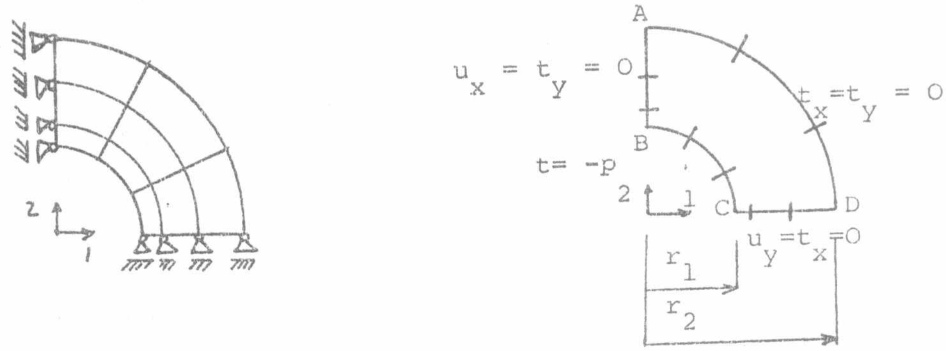
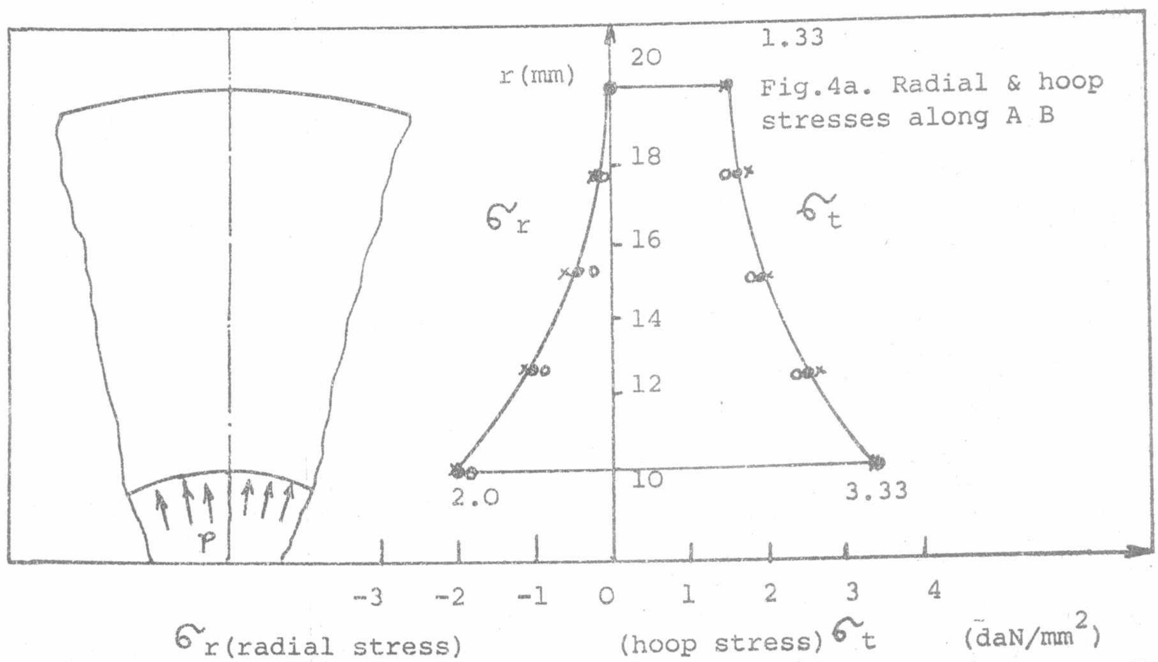
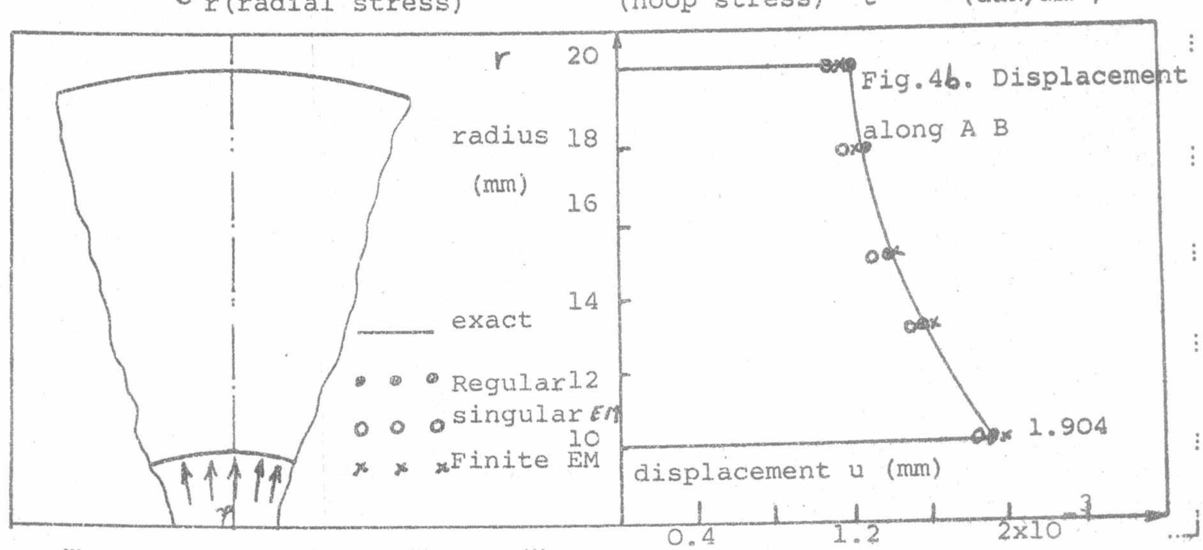


b) Finite element discretization a)Boundary element discretization

Figure 3. Thick walled cylinder - discretization.





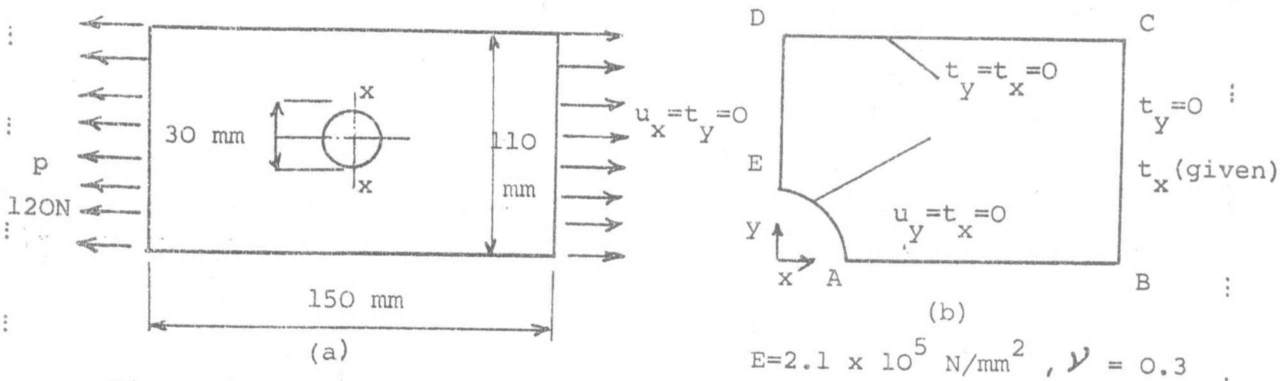


Figure 5.a. Plate with a circular hole under uniform tension

Figure 5.b. Boundary conditions for quarter domain.

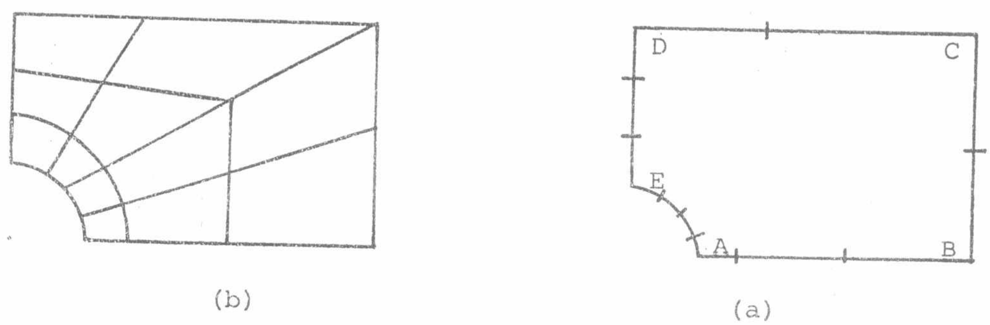


Figure 6. Boundary and Finite Element discretization.

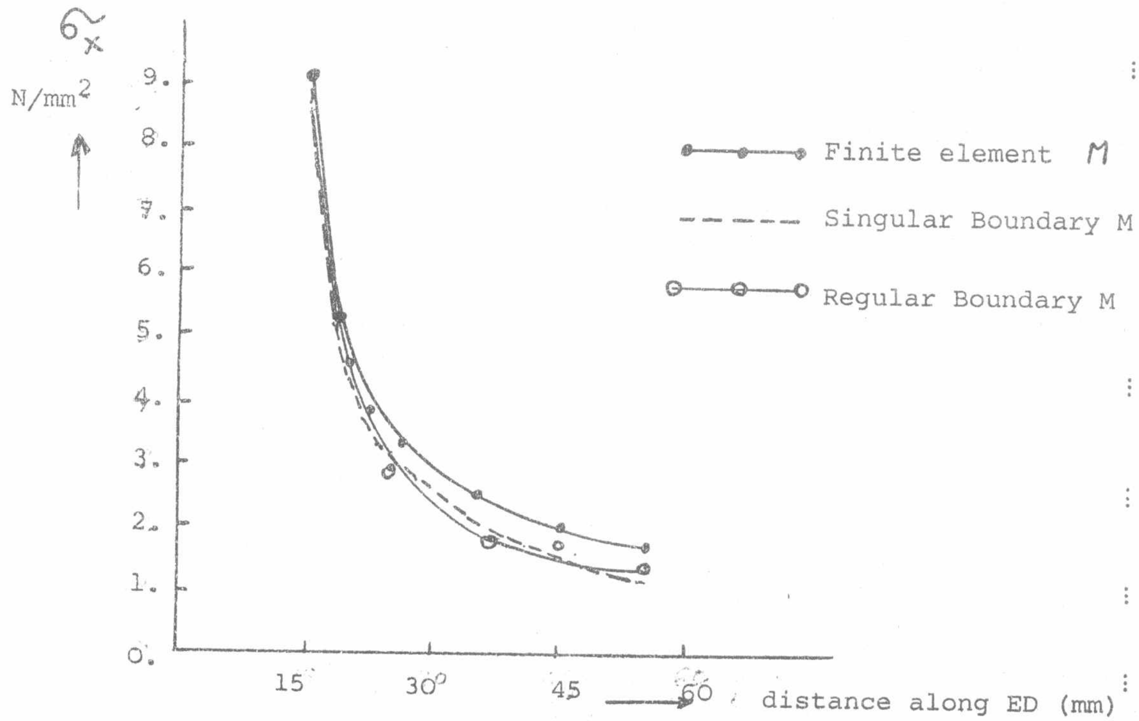


Figure 7. Stress distribution along edge ED of the plate using quadratic elements.

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Stress Concentration in a thin plate under uniform Tension due to a small Circular Hole:

Thin plate with a circular hole in the center is subjected to a uniform tensile load as shown in Figure 5a. The diameter of the hole is taken as less than one third of the width of the plate to produce high stress concentration at points xx. Only a quarter of the plate needs to be considerd because of symmetry, and is subject to boundary conditions shown in Figure 5b.

The problem (plane stress) was solved by dividing the boundary into 14 quadratic curved elements. The Finite element discretization was 12 quadratic elements as shown in Figure 6a,b. The curves showing stress distribution along DE are plotted in Figure 7.

DISCUSSIONS and CONCLUSIONS

A regular boundary element method for use with two dimensional stress analysis problems has been presented. This method has been applied using continuous (conventional) boundary elements having quadratic variation of the field quantity suported by an 3-node isoparatric quadrilateral geometric element. A subsidiary investigation showed that the singularity of the fundamental solution kernel function was 'best' located along outward normal from a freedom node at a distance approximately equal to the shortest internodal distance within the element considered.

Results obtained , using the quadratic variant of boundary element, for two test problems have been presented. The first, thick walled cylinder under internal pressure, was chosen because its solution is well known. Principally, it was chosen as a validity test on the coding. The mutual agreement for the Regular Boundary Element Method is around $\pm$ 0.06 units.

The second, plate perforated with a circular hole. Here the normal stresses along the line ED, Figure 7, as given by the singular and regular methods for quadratic boundary elements were both in good mutual agreement and agreed well with the finite element result. It is noted that the boundary element stresses are consistently below the finite element values.

In conclusion- the regular boundary element method presented here has given accurate results for problems examined, the presence of a singularity in the exact solution can be tolerated without the necessity of highly refined meshes near the singularity in that good solution away from the singularity can be obtained.

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