MILITARY TECHNICAL COLLEGE

CAIRO - EGYPT

SIMPLIFIED ANALYSIS OF THICK LAMINATED FIBROUS COMPOSITES USING A STATICAL APPROACH

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ABSTRACT

Reissner's plate theory assumptions are chosen to approximate the stress field in which the longitudinal stresses are assumed to vary linearly along the plate thickness. For balanced, symmetric, laminates, a stiffness may be taken as the linear average of the corresponding ply stiffnesses. In that case, only 4 laminate stiffness moduli with simple stress-strain relations can be utilized. Using the average laminate properties and following the variational analysis, the assumed stress functions and the corresponding displacements can be obtained. As a numerical example, we consider the problem of cylindrical bending of a symmetric cross-ply laminate, consists of 3 layers of graphite/epoxy unidirectional composites. For relatively thick laminates, where the classical plate theory gives a very poor estimate, our results are in a good agreement with the exact elasticity solution, specially for the deflection and transversal stresses.

INTRODUCTION

Laminated fibrous composites are a hybrid class of composites involving both fibrous composites and lamination techniques. The classical laminated plate theory passesses certain deficiencies, it has wide usage as the basis for the analysis and design of structural laminates. Among the characteristics of the classical theory, which limit its generality in the description of relatively thick laminates, is the neglect of transverse shear and normal deformations. The general three dimensional analysis of laminated composites presents a formidable task. Hence, the majority of investigators have utilized approximate simplified analysis or numerical techniques in which the displacement functions are assumed along the laminate thickness. Pagano [4],[5] presented the exact analysis of a laminated plate under cylindrical bending. Whitney [8] introduced an approximate method to incorporate the influence of shear deformation on plate deflection in composite laminates as well as displacements.

In the present analysis, we shall present a simplified procedure depending on a statical approach to deal with the problem of thick composite laminates. The author in previous work [1] introduced a generaliztion of

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classical Reissner theory to deal with anisotropic materials and he presented an application to thick anisotropic plates. The stress field, through the laminate thickness, may be chosen to follow the same assumptions as in the thick anisotropic plate. The longitudinal stresses $\sigma_{\alpha\beta}$ are assumed to be linearly distributed across the thickness. Using the equilibrium equations, the transverse shear stresses $\sigma_{\alpha 3}$ and the transverse normal stress σ_{33} can be expressed as functions of the out of plane variable x_3. The simplest laminated composites consist of 0 and 90 degrees and balanced plies; i.e., the positive and negative off-axis ply orientations appear in pairs. This class of laminates is orthotropic, hence the laminate is characterized by only four stiffness elastic moduli. The resulting stress-strain relations are identical to those for the isotropic material except the number of moduli are increased from two to four.

In a racent work, Tsai [6] showed that the stiffness moduli of an orthotropic laminate are simply the linear averages of the corresponding ply stiffnesses, i.e., the rule-of-mixture relation between the stiffness of the ply and the laminate is applicable. He also indicated that the laminate and the plies have the same three invariants limiting the stiffness moduli. We will limit ourselves to deal with class of orthotropic, balanced, laminates. Considering the average stiffness moduli of the laminate, the statical variational analysis [1] leads to equations of equilibrium and Euler's equations which permit the determination of the unknown stress functions and the corresponding displacements. It may be indicated that weighted averages of displacements, rather than displacements themselves, are obtained.

THEORITICAL APPROACH

Let us consider a laminate composed of (m) orthotropic layers such that the various axes of material symmetry are parallel to the plate axes x_i. If the laminate is loaded on the upper and lower faces by the two loads q_+ and q_- respectively as indicated in Fig.1., the total load q and the mean extensional load p may be written as

$$q = q_+ + q_- ,$$
$$p = (q_+ - q_-)/2. \tag{1}$$

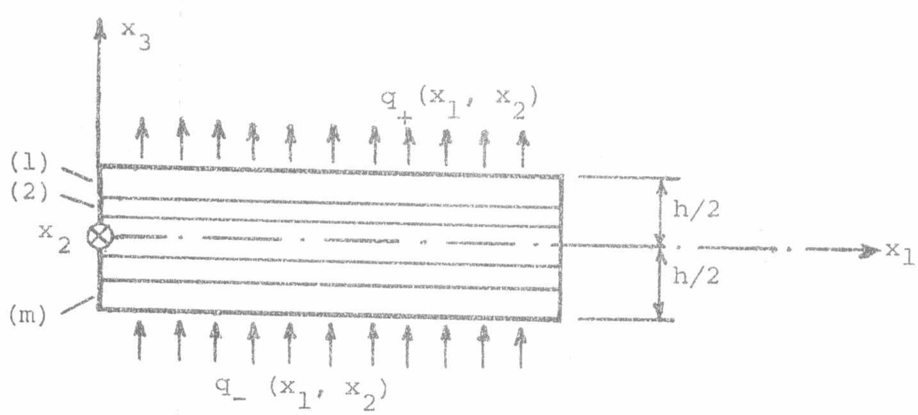

Fig.1. Laminate notation and loading.

As an approximation to the true stress field through the plate thickness, the longitudinal stresses $\sigma_{\alpha\beta}$ are assumed to be [1] ,

$$\sigma_{\alpha\beta} = \frac{1}{h} N_{\alpha\beta}(x_1,x_2) + \frac{12}{h^3} x_3 \, M_{\alpha\beta}(x_1, x_2) \tag{2}$$

where $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are membrane forces and bending moments respectively. To satisfy the statical boundary conditions on the two laminate faces and using the stress equilibrium equations, the transverse shear stresses $\sigma_{\alpha 3}$ and the transverse normal stress σ_{33} are chosen as;

$$\begin{aligned} \sigma_{\alpha 3} &= \frac{3}{2h} \left(1 - 4 \frac{x_3^2}{h^2}\right) S_\alpha (x_1, x_2) \\ \sigma_{33} &= p(x_1, x_2) + \frac{1}{2}\left(3 \frac{x_3}{h} - 4 \frac{x_3^3}{h^3}\right) q \,(x_1, x_2) \end{aligned} \tag{3}$$

where S_α are the shearing forces.

If the above stress expressions are substituted into the stress equilibrium equations ;

$$\sigma_{ij,j} = 0 \tag{4}$$

then the stress resultants $N_{\alpha\beta}$, $M_{\alpha\beta}$ and S_α must satisfy the following equilibrium equations

$$\begin{aligned} N_{\alpha\beta,\beta} &= 0 \\ M_{\ \beta,\beta} - S_\alpha &= 0 \\ S_{\alpha,\alpha} + q &= 0 \end{aligned} \tag{5}$$

Let $Q^{(n)}_{\alpha\beta\gamma\delta}$ be the reduced stiffness coefficients of plane stress for the n-th layer or ply group. These stiffness coefficients may be expressed as:

$$Q^{(n)}_{\alpha\beta\gamma\delta} = C^{(n)}_{\alpha\beta\gamma\delta} - C^{(n)}_{\gamma\delta 33} \, C^{(n)}_{\alpha\beta 33} / C^{(n)}_{3333} \tag{6}$$

where C_{ijkl} are the elastic moduli for the n-th layer defined by the generalized Hooke's law in the form

$$\sigma_{ij} = C_{ijkl} \, \varepsilon^{kl} \tag{7}$$

If the thickness, locations, and material properties of the laminate are symmetric about the middle surface $x_3 = 0$, the laminate becomes orthotroic. In this case of lamination, the laminate stiffness may be considered as the linear average of the ply stiffness [6], i.e.,

$$\begin{aligned} \bar{Q}_{\alpha\beta\gamma\delta} &= \frac{1}{h} \sum_{n=1}^{m} Q^{(n)}_{\alpha\beta\gamma\delta} \, h_{(n)} \\ &= \sum_{n=1}^{m} Q^{(n)}_{\alpha\beta\gamma\delta} \, v_{(n)} \end{aligned} \tag{8}$$

where $h_{(n)}$ = thickness of the n-th layer or ply group

$v_{(n)}$ = volume fraction of the n-th layer.

It may be noted that, for case of orthotropic, symmetric, laminate, there is no coupling between the normal stresses and shear deformations and we have only four laminate stiffness moduli $\bar{Q}_{1111}$, $\bar{Q}_{2222}$, $\bar{Q}_{1122}$ and $\bar{Q}_{1212}$.

The variational analysis depending on the assumed stress field, Equations (2) and (3), leads to the following constitutive relations [1] which are organized to suit the orthotropic laminate conditions:

$$N_{\alpha\beta} = h\,(\bar{Q}_{\alpha\beta\gamma\delta}\, u_o^{\gamma,\delta} + \bar{B}_{\alpha\beta}\, p)$$
$$M_{\alpha\beta} = \frac{h^3}{12}\,(\bar{Q}_{\alpha\beta\gamma\delta}\, \psi^{\gamma,\delta} + \frac{6}{5h}\,\bar{B}_{\alpha\beta}\, q) \qquad (9)$$
$$S_{\alpha} = h\,\bar{d}_{\alpha\beta}(\psi^{\beta} + w_o^{,\beta})$$

where

$$B_{\alpha\beta} = C_{\alpha\beta 33} \,/\, C_{3333}$$
$$d_{\alpha\beta} = \frac{5}{6}\, C_{\alpha 3\beta 3} \qquad (10)$$

u_{α}^{o}, ψ_{α} and w are the weighted average displacements defined by the integrals

$$\{u_{\alpha}^{o}, w_o\} = \frac{3}{2h}\int_{-h/2}^{h/2} \{u_{\alpha}, w\}(1 - 4\frac{x_3^2}{h^2})\, dx_3$$
$$\qquad (11)$$
$$\psi_{\alpha} = \frac{12}{h^3}\int_{-h/2}^{h/2} u_{\alpha} x_3 \quad dx_3$$

the over barred quantities represents average properties for the laminate which are based on rule-of-mixtures relations, as seen in Equation (8). The five equilibrium Equations (5) and the eight constitutive relations (9) form a set of 13 equations the solution of which leads to the determination of the 13 unknown functions $N_{\alpha\beta}$, $M_{\alpha\beta}$, Q_{α}, u_{α}^{o}, ψ_{α} and w_o.

NUMERICAL RESULTS AND DISCUSSION

As a numerical example, we will treat the problem of an orthotropic laminate under cylindrical bending and acted upon by a sinusoidal load. The exact elasticity solution for this problem was presented by Pagano [4]. In order to compare our results to the corresponding exact solutions, we consider layers of symmetric unidirectional fibrous composite material possessing the following Engineering stiffness properties, which simulate a high modulus graphite / epoxy composite;

$$E_{11} = 175 \text{ GPa}, \quad G_{12} = 3.5 \text{ GPa}, \quad \nu_{12} = 0.25$$
$$E_{22} = 7.1 \text{ GPa}, \quad G_{13} = 3.5 \text{ GPa}, \quad \nu_{13} = 0.25 \qquad (12)$$
$$E_{33} = 7.1 \text{ GPa}, \quad G_{23} = 1.4 \text{ GPa}, \quad \nu_{23} = 0.25$$

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where index 1 signifies the direction parallel to the fibers, indices 2 and 3 signify the longitudinal and transverse directions and ν_{ij} is the Poisson's ratio, measuring strain in the jth direction under uniaxial normal stress in the i-th direction.

Three cross ply laminate is considered. The fiber axes of the layers make, with the laminate axis 1, the angles of $0^{o}/90^{o}/0^{o}$, Fig.2. The results are obtained for various span to depth ratio, r=a/h. As a particular consideration a thick laminate with span-to-depth ratio of 4 is selected for detailed analysis.

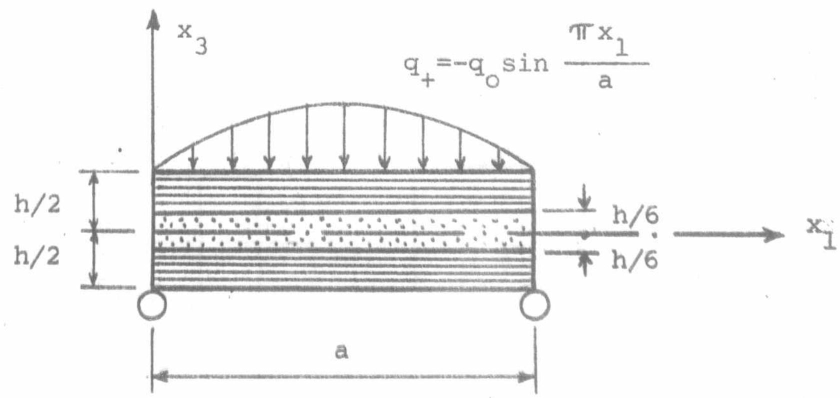

Fig.2. Three-ply laminate under cylindrical bending

The reduced stiffness coefficients $Q_{\alpha\beta\gamma\delta}$, the shear rigidity coefficients $d_{\alpha\beta}$ and the factors $B_{\alpha\beta}$ for the layers can be calculated using the given engineering properties (12). Using the rule-of-mixtures relations, the corresponding average laminate properties are obtained as :

$$\bar{Q}_{1111} = 119.33 \text{ GPa} , \quad \bar{d}_{11} = 2.33 \text{ GPa} ,$$
$$\bar{Q}_{2222} = 63.26 \text{ GPa} , \quad \bar{d}_{22} = 1.75 \text{ GPa} ,$$
$$\bar{Q}_{1122} = 1.78 \text{ GPa} , \quad \bar{B}_{11} = 0.293 \quad (13)$$
$$\bar{Q}_{1212} = 3.50 \text{ GPa} , \quad \bar{B}_{22} = 0.273$$

Our trend is to indicate by plotting the comparison between the obtained results and the corresponding exact solutions. The following normalized quantities are defined in connection with all curves in Fig.3:

$$\bar{\sigma}_{11} = \sigma_{11}(\tfrac{a}{2}, x_3)/q_o , \quad \bar{u}_1 = 10\, E_{22} u_1(a , x_3)/q_o h,$$
$$\bar{\sigma}_{33} = \sigma_{33}(\tfrac{a}{2}, x_3)/q_o , \quad \bar{w}_o = 100 E_{22} h^3 w_o (\tfrac{a}{2})/q_o a^4, \quad (14)$$
$$\bar{\sigma}_{13} = \sigma_{13}(a , x_3)/q_o , \quad r = \frac{a}{h} , \quad \bar{x}_3 = \frac{x_3}{h} .$$

In the various curves, the solid line indicates the elasticity solution [4], the result of the classical plate theory (CPT) is denoted by a dash and point while the dashed line represents our result.

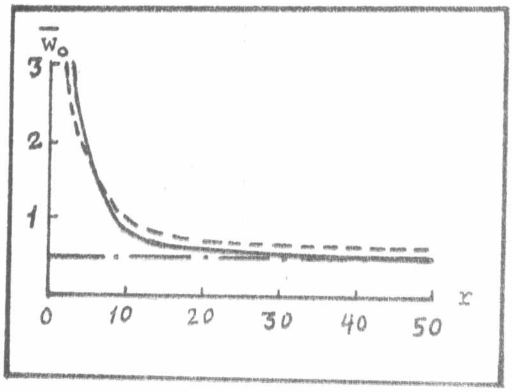

Fig. 3 (a)

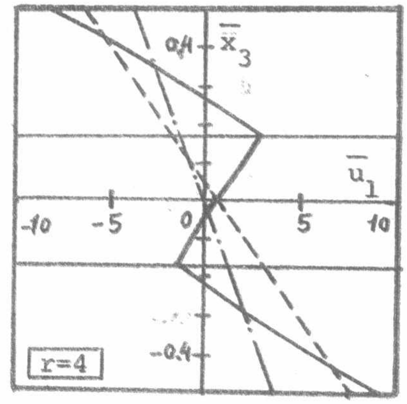

Fig.3 (b)

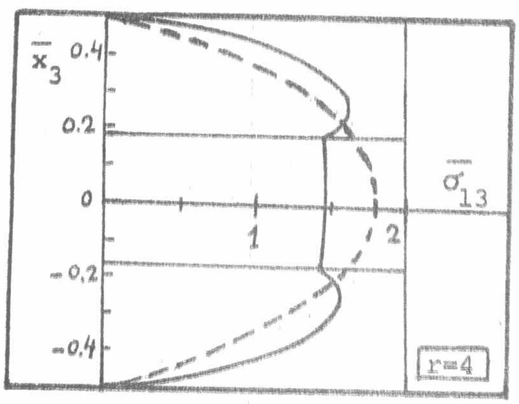

Fig. 3. (c)

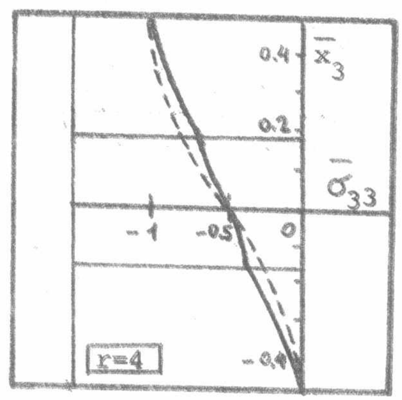

Fig.3 (d)

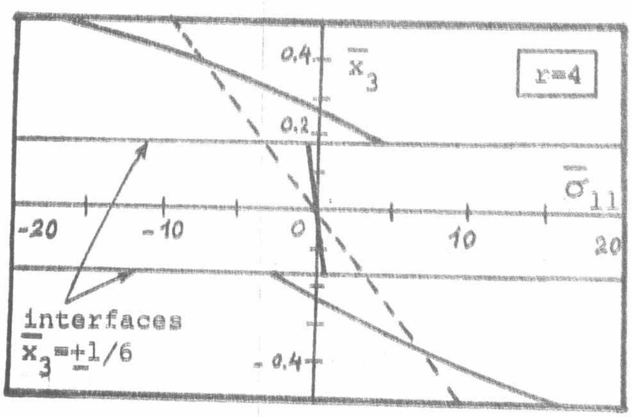

Fig. 3. (e)

——— Exact Solution [4]

– – – Present Analysis

—·— Classical plate theory (CPT)

Fig.3. Comparison of our results with exact and CPT solutions for average displacements and stresses of 3 cross-ply laminate.

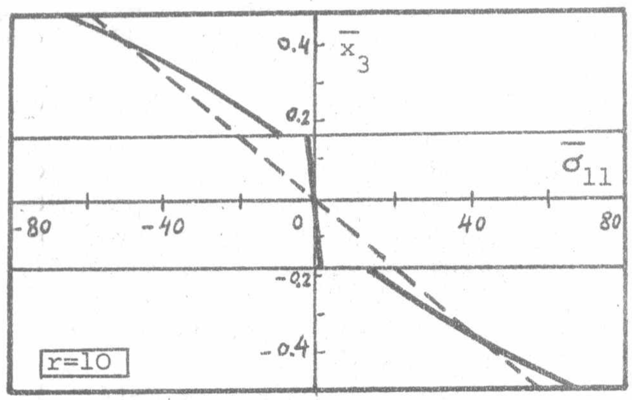

Fig. 3. (f)

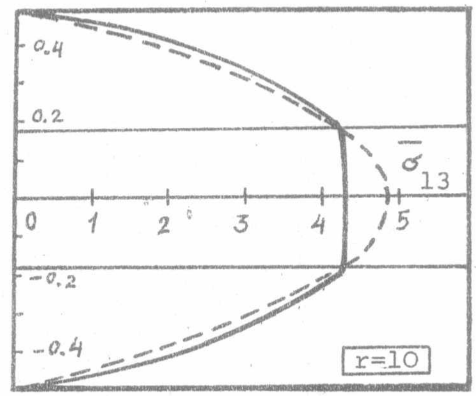

Fig. 3. (g)

Fig. 3 (a) shows the relationship of maximum deflection $\bar{w}$ to r. As would be expected, the CPT underestimates the plate deflection and gives a very poor estimate for relatively thick plates, low values of r. Fig.3 (b) shows the variation of the longitudinal displacement $\bar{u}_1$ along the laminate thickness for r = 4. The present approximate analysis underestimates the maximum value of $\bar{u}_1$ on the laminate faces, the error is approximately 17%. The error decreases with higher values of r. At the middle plane, $\bar{x}_3=0$, the value of longitudinal displacement is not equal to zero as obtained from CPT but it has a very small value due to the effect of symmetric part of the load.

Fig.3 shows the distribution of longitudinal stress $\bar{\sigma}_{11}$, transverse normal stress $\bar{\sigma}_{33}$ and transverse shear stress $\bar{\sigma}_{13}$ through the laminate depth. The differences between our results and the exact solution decrease with higher values of r.

CONCLUSION

We have presented an approximate analysis for symmetric laminates consisting of orthotropic layers. Average stiffness moduli are considered to characterize the laminate properties. The cylindrical bending of a laminate under sinosidal loading is treated, as numerical example. The obtained results are compared with solutions given in literatures. The assumption of continuity of longitudinal stresses at the laminate interfaces violates with the continuity of displacements there. Hence, the signficiant differences between our results and the exact elasticity solutions occur at the interfaces between layers. These differences decrease as the laminate becomes thinner. For transverse stresses, which are neglected in the CPT, our results are in a good agreement with the exact solutions. These stresses are responsible for delamination failures. In the present simplified approach it may be noted that the results for the out-of-plane displacement and the tranverse stresses are relatively better than the longitudinal displacements and stresses.

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NOMENCLATURE

a, h	laminate span and depth
C_{ijkl}	generalized elastic moduli
d	shear elastic moduli
$\bar{d}$	laminate average shear elastic moduli.
E_{ii}	engineering elastic moduli
G_{ij}	engineering shear moduli
$i,j,k,l = 1,2,3$...	three dimensional tensor indices
$M_{\alpha\beta}$	bending moments
$N_{\alpha\beta}$	membrane forces
$Q_{\alpha\beta\gamma\delta}$	plane stress elastic moduli
$\bar{Q}_{\alpha\beta\gamma\delta}$	laminate average stiffness moduli
q_+, q_-, q, p	lateral loads
r	span-to-depth ratio
S_α	shearing forces
u_i	displacement vector
u^o_α	average in-plane displacements
w	average laminate deflection
x_i	cartesian coordinates
$,i$	derivative with respect to x_i
$\alpha,\beta,\gamma,\delta = 1,2$ ...	two dimensional tensor indices
ν_{ij}	Poisson's ratios
ψ_α	average inclination angles.