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BASIC CONCEPTS IN MODAL ANALYSIS

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ABSTRACT

Modal analysis is the presentation of the dynamic properties of a system in terms of the contributions due to the independent modes of vibration of the system. This method of analysis is powerful and has become very useful in the analysis of experimental test data due to recent developments in real time frequency analysis and digital data processing. This paper outlines the recent developments of the method with main emphasis is placed on the use of this approach in analysing the dynamic performance of complex systems and in solving their design problems.

INTRODUCTION

Modal analysis is the technique of measuring the general vibration of a structure [1,2] to determine its characteristic modes of vibration. Each mode of vibration and its associated natural frequency are unique properties of the structure. The value of modal analysis lies in its ability to determine the relative motion of points on the structure when a resonant frequency is excited. With this information, a redesign of the structure becomes possible such that the problem mode of vibration is correctly taken into account.

This paper describes the theoretical foundations and assumptions underlying most of the minicomputer-based Modal Analysis systems. These systems allow the determination of modes of vibration directly from vibration measurements and provide dynamic modeling and design synthesis based on this vibration data.

EQUATIONS OF MOTION

The number of coordinates required to describe the vibration of the structure as well as the number of experimental res-

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ponse measurements needed from the structure is known as the number of degrees of freedom n. Considering matrix notation, the equations of motion for a linear system take the form

$$[M]\,\ddot{\underline{x}}(t) + [C]\,\dot{\underline{x}}(t) + [K]\,\underline{x}(t) = \underline{f}(t) \qquad (1)$$

$\underline{f}(t)$ = vector of input forces

$\underline{x}(t)$ = dynamic response vector

$[M]$ = mass matrix

$[C]$ = matrix of damping coefficients

$[K]$ = matrix of stiffness coefficients.

The vector **f**(t) represents the time varying force load applied to each degree of freedom of the structure. In an experimental situation it is often convenient to excite the structure at only one point, in one direction. The mass, stiffness and damping matrices [M], [K], and [C] are all assumed to be n x n, symmetrical and non-singular. The stiffness and damping matrices in general have off diagonal enteries which provide coupling between coordinates. It is a fundamental result of the study of linear differential equations like (1) that this coupling arises only from the choice of coordinates $\underline{x}$ used to describe the model. Another set of coordinates exists which, if used instead of $\underline{x}$, would yield a set of n individual equations each with one unknown, rather than the coupled set of equations (1) (if the damping matrix has a special form). The dynamic response of each of these special coordinates is given by the single mass, spring and damper equation of motion:

$$m\,\ddot{q}(t) + c\,\dot{q}(t) + k\,q(t) = f(t) \qquad (2)$$

$f(t)$ = input force

$q(t)$ = dynamic response

m = mass

c = damping coefficient

k = stiffness coefficient.

For a general symmetric damping matrix, equation (1) may still be decoupled with the proper choice of 2n special coordinates [3]. The two cases of damping matrices are referred to as "proportional" or "Rayleigh" damping and "non-proportional" damping. The special coordinates are, of cource, the modal coordinates of the structure. The oscillation of each modal coordinate is completely independent of all the other coordinates. They may be considered separate one-degree of freedom systems, each with its own natural frequency. This property of the modal coordinates is known as the "orthogonality" of mode shapes. The modal coordinates $\underline{q}$ of a structure are related to the original coordinates $\underline{x}$ by the equation

$$\underline{q} = [A]\ \underline{x}. \tag{3}$$

The rows of [A] dictate the amount of each element of $\underline{x}$ which forms one modal coordinate, for example

$$q_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n. \tag{4}$$

The vector $\underline{a}_i^T$ is called a mode shape vector. It is the goal of modal analysis to determine these vectors. The matrix [A] is determined by the requirement that it decouple equation (1). The first step is to invert equation (3), substitute for $\underline{x}$ in (1), and multiply both sides by [A] to obtain:

$$[A][M][A]^{-1}\ddot{\underline{q}} + [A][C][A]^{-1}\dot{\underline{q}} + [A][K][A]^{-1}\underline{q} = [A]\ \underline{f}(t)$$

This set of equations will consist of independent single degree of freedom differential equations only if the matrix products $[A][M][A]^{-1}$, $[A][C][A]^{-1}$, and $[A][K][A]^{-1}$ result in diagonal matrices. No single matrix [A] can diagonalize three general matrices [M], [C] and [K]; [4]. However, two of the three may be diagonalized, say [M] and [K].

PROPORTIONAL DAMPING

If the third matrix [C] is a linear combination of [M] and [K] it too will be diagonalized by [A].

$$[C] = \alpha[M] + \beta[K] \tag{6}$$

where α and β are arbitrary real numbers. An important property of [A] is that for real and symmetric matrices [M] and [K], it can be normalized such that its transpose is equal to its inverse, that is

$$[A]^T = [A]^{-1}. \tag{7}$$

Therefore, an orthogonal matrix [A] exists such that equation (5) takes the following diagonalized form:

$$D(\bar{m}_i)\ddot{\underline{q}} + D(\bar{c}_i)\dot{\underline{q}} + D(\bar{k}_i)\underline{q} = [A]\underline{f}(t). \tag{8}$$

where $D(\bar{c}_i) = \alpha\, D(\bar{m}_i) + \beta\, D(\bar{k}_i)$. (9)

It is also possible to determine the response of the structure frequency by frequency. A typical equation of (8) with a sinusoidal forcing function is written as

$$\bar{m}_i \ddot{q}_i + \bar{c}_i \dot{q}_i + \bar{k}_i q_i = \underline{a}_i^T\, \underline{F}(\omega) e^{j\omega t} \qquad (10)$$

where $\underline{F}(\omega)$ is the vector of Fourier transforms calculated from $f(t)$. Equation (10) has the well-known solution:

$$q_i(t;\omega) = \frac{\underline{a}_i^T\, \underline{F}(\omega)}{(\bar{k}_i - \omega^2 \bar{m}_i) + j(\omega \bar{c}_i)}\, e^{j\omega t}. \qquad (11)$$

This is the response of the modal coordinate q_i to a sinusoidal force with frequency ω. The vector $\underline{F}(\omega)$ in general has complex elements, and equation (11) can be written as

$$q_i(t;\omega) = \frac{|\underline{a}_i^T\, \underline{F}(\omega)|}{\sqrt{(\bar{k}_i - \omega^2 \bar{m}_i)^2 + (\omega \bar{c}_i)^2}}\, e^{j[\alpha - \beta + \omega t]} \qquad (12)$$

where

$$\alpha(\omega) = \tan^{-1}\left(\frac{\underline{a}_i^T\, Im\{\underline{F}(\omega)\}}{\underline{a}_i^T\, Re\{\underline{F}(\omega)\}}\right), \text{and } \beta(\omega) = \tan^{-1}\left(\frac{\omega \bar{c}_i}{\bar{k}_i - \omega^2 \bar{m}_i}\right).$$

The real response of this coordinate q at ω is the real part of (12) times 2 where the factor 2 reflects the contribution of $F(-\omega)$. An excitation at the particular frequecy ω_o defined by:

$$\omega_o = \left(\frac{\bar{k}_i}{\bar{m}_i}\right)^{1/2} \qquad (13)$$

has the real response

$$q_i(t;\omega_o) = \frac{2|\underline{a}_i^T\, \underline{F}(\omega)|}{\omega_o \bar{c}_i} \cos\left(\alpha(\omega_o) - \frac{\pi}{2} + \omega_o t\right) \qquad (14)$$

Notice that this response lags behind the force by 90°. Also, we see that the magnitude of q_i increases as its damping $\bar{c}_i$ decreases. Undamped, this coordinate will theoretically

6

make infinitely large excursions. This is known as resonance, and ω_o is the resonant frequency of q_i.

The total response of q_i to the general excitation $\underline{a}_i^T \underline{f}(t)$ is obtained by integrating its frequency response, equation (12), over all frequencies

$$q_i(t) = \int_{-\infty}^{\infty} \frac{\underline{a}_i^T \underline{F}(\omega) e^{j\omega t}}{(\bar{k}_i - \omega^2 \bar{m}_i) + j(\omega \bar{c}_i)} d\omega. \tag{15}$$

This equation can be viewedas the inverse Fourier transform of a product of two Fourier transforms. One factor is $\underline{a}_i^T \underline{F}(\omega)$ and the other is

$$H_i(\omega) = \frac{1}{\bar{k}_i - \omega^2 \bar{m}_i + j(\omega \bar{c}_i)} \tag{16}$$

The convolution theorem [5] says that in this situation, equation (12) is equal to the convolution of $\underline{a}_i^T \underline{f}(t)$ and $h_i(t)$ given by

$$q_i(t) = \int_{-\infty}^{\infty} \underline{a}_i^T \underline{f}(\tau) h_i(t-\tau) d\tau \tag{17}$$

where $$h_i(t) = \int_{-\infty}^{\infty} H_i(\omega) e^{j\omega t} d\omega. \tag{18}$$

This function $h_i(t)$, the inverse transform of $H_i(\omega)$, is called the impulse response of q_i. It is the total response of q_i to a unit impulse excitation, $\delta(t)$.

The solution of equation (18) for the modal coordinate q_i is given by

$$h_i(t) = \frac{e^{-\sigma t}}{2j\omega_d}\left[e^{j\omega_d t} - e^{-j\omega_d t}\right] = \frac{e^{-\sigma t}}{\omega_d} \sin(\omega_d t) \tag{19}$$

where $$\sigma = \frac{\bar{c}_i}{2\bar{m}_i} \quad \text{and} \quad \omega_d = \sqrt{\frac{\bar{k}_i}{\bar{m}_i} - \left(\frac{\bar{c}_i}{2\bar{m}_i}\right)^2}. \tag{20}$$

The impulse response $h_i(t)$ and its Fourier transform $H_i(\omega)$ are important functions representing the dynamic characteristics of a linear system. Equation (16)(or (12)) can be viewed as a process which uses the impulse response to convert the force input into a response output. The function $H_i(\omega)$ is called the transfer function of the modal coordinate q_i. The functions $h_i(t)$ and $H_i(\omega)$ are basic properties of a linear system and can be measured[1] either as the response to an impulse or as the ratio of the Fourier transforms of measured input and response. Experimental modal analysis uses the techniques of signal processing to determine the Fourier transform of applied forces and the structural response. To determine the transfer functions in terms of the original coordinates $\underline{x}$, we use equation (3) which transforms $\underline{x}$ into $\underline{q}$. The matrix $[\bar{A}]$ also transforms $\underline{X}(\omega)$, the Fourier transform of $\underline{x}(t)$, into $\underline{Q}(\omega)$:

$$\underline{Q}(\omega) = [A]\, \underline{X}(\omega). \tag{21}$$

Considering the Fourier transform of $q_i(t)$, obtained from (12), the resulting n equations can be collected into the one matrix equation:

$$\underline{Q}(\omega) = D(H_i(\omega))\, [A]\, \underline{F}(\omega) \tag{22}$$

where $D(H_i(\omega))$ is a diagonal matrix. Therefore the matrix equation which relates forces applied to the original coordinates and their response is

$$\underline{X}(\omega) = [A]^T\, D(H_i(\omega))\, [A]\, \underline{F}(\omega) \tag{23}$$

The matrix $[H(\omega)]$, defined as

$$[H(\omega)] = [A]^T\, D(H_i(\omega))\, [A], \tag{24}$$

is the matrix of transfer functions which can be related to direct measurements of the dynamic structural response. A typical element $h_{ij}(\omega)$ of $[H(\omega)]$ is the Fourier transform of the output at coordinate x_i divided by the Fourier transform of the input force at x_j. Also, experimental measurements at the coordinates $\underline{x}$ are usually made in terms of output accelerations[6].

Thus, $$\underline{Acc}(\omega) = -\omega^2 [H(\omega)]\, \underline{F}(\omega). \tag{25}$$

It is this transfer function matrix, $-\omega^2[\tilde{H}]$ which is usually obtained in experimental modal surveys. It is important to distinguish the measured transfer function $-\omega^2[\tilde{H}]$ from its theoretical counterpart $-\omega^2[H]$ in equation (24). The matrix $[\tilde{H}]$ is obtained from experimental data and represents the real characteristics of the structure at certain, hopefully representative, points.

Each element of the transfer function matrix $D(H_i(\omega))$ given by equation (15) can be factored and then expanded [1] by partial fractions into:

$$H_i(\omega) = \frac{\dfrac{1}{2j\omega_d\, \bar{m}_i}}{[j\omega - (-\sigma + j\omega_d)]} + \frac{\dfrac{-1}{2j\omega_d\, \bar{m}_i}}{[j\omega - (-\sigma - j\omega_d)]} \tag{26}$$

The term ω_d is the damped natural frequency for this modal coordinate and σ is the inverse of its damping time constant. These two parameters are defined in terms of the i-th modal mass, stiffness and damping by equations (19) and (20). The complex number $p_i = -\sigma + j\omega_d$, is called a pole of the transfer function. Define

$$A_i = \frac{1}{2j\omega_d \bar{m}_i} \quad (27)$$

The coefficient A_i is called the residue at the pole p_i. In the notation for complex poles and residues, equation (26) takes the form

$$H_i(\omega) = \frac{A_i}{j\omega - p_i} + \frac{A_i^*}{j\omega - p_i^*} \quad (28)$$

where the star denotes complex conjugation.

It is the purpose of parameter estimation techniques in modal analysis to define the elements $\bar{m}_i$, $\bar{c}_i$, and $\bar{k}_i$ and the matrix [A] which best fit [H] to [$\hat{H}$].

The value $\bar{c}_i = 2\sqrt{\bar{k}_i \bar{m}_i}$ is called critical damping. When the modal damping $\bar{c}_i$ is less than this critical value, ω_d is real and p_i is complex. When $\bar{c}_i$ is greater than critical damping, the damped natural frequency becomes the imaginary number, $j|\omega_d|$. Equation (28) is still valid, however, there are now two distinct real poles

$$p_i = -\sigma + |\omega_d| \qquad p_i^* = -\sigma - |\omega_d| \quad (29)$$

with equal but opposite in sign real residues A_i and A_i^*.

It is also clear from equation (24) that the transfer function is the sum of symmetric matrices and is symmetric.

$$h_{ij} = \sum_{k=1}^{n} H_k a_{ki} a_{kj} \quad (30)$$

Symmetry of [H] implies that $h_{ij}=h_{ji}$ or that the response of x_i to a force at x_j is exactly the same as the response of x_j to the same force at x_i. This fact is useful in experimental transfer function measurements. Incorporating the complex pole and residue form of H_k given by (27) into (30) we obtain

$$h_{ij} = \sum_{k=1}^{n} \frac{A_k a_{ki} a_{kj}}{j\omega - p_k} + \frac{A_k^* a_{ki} a_{kj}}{j\omega - p_k^*} \quad (31)$$

There are 2n complex constants (p_k, A_k a_{ki} a_{kj}) which are adjusted by curve fitting algorithms until the transfer function h_{ij} are approximately equal to the measured $\tilde{h}_{ij}$.

NON-PROPORTIONAL DAMPING

In order to handle general damping matrices, a 2n x 1 vector $\underline{z}$ is defined with its first n elements equal to $\dot{\underline{x}}$ and its last n equal to $\underline{x}$. The n equations of motion (1) can be written in terms of these new coordinates $\underline{z}$

$$[R]\ \dot{\underline{z}}(t) + [S]\ \underline{z}(t) = \underline{g}(t) \tag{32}$$

where

$$[R] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix},\ [S] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix},\text{and}\ \underline{g}(t) = \begin{Bmatrix} \underline{0} \\ \underline{f}(t) \end{Bmatrix}.$$

The matrices [R] and [S] are real and symmetric and can be diagonalized by a 2n x 2n matrix [A]. It is interesting to note that neither [R] nor [S] is positive definite, therefore, [A] is a complex rather than real matrix. The rows of [A] occur in complex conjugate pairs and [A] can be made orthonormal. The 2n x 1 modal coordinate vector $\underline{q}$ is related to $\underline{z}$ by the equation

$$\underline{q} = [A]\underline{z} \tag{33}$$

and (5) takes the form

$$[A][R][A]^T\dot{\underline{q}} + [A][S][A]^T\underline{q} = [A]\underline{g}. \tag{34}$$

Since [R] and [S] can be simultaneosly diagonalized, we obtain the diagonal matrices $D(\bar{r}_i)$ and $D(\bar{s}_i)$, where the coefficients $\bar{r}_i$ and $\bar{s}_i$ in this case are complex numbers (which occur in complex conjugate pairs). Notice that since the first n entries of $\underline{g}$ are all zero, the matrix products $\underline{a}_i^T\ \underline{g}$ may be simplified by introducing the 1 x n row vector $\underline{b}_i^T$ which simply consists of the last n elements of $\underline{a}_i^T$, thus

$$\underline{a}_i^T\ \underline{g} = \underline{b}_i^T\ \underline{f}. \tag{35}$$

Equation (34) has the well known solution:

$$q_i(t;\omega) = \frac{\underline{b}_i^T\ \underline{F}(\omega)\ e^{j\omega t}}{(j\omega\ \bar{r}_i + \bar{s}_i)}. \tag{36}$$

6

and equations (16) and (22) take the form

$$H_i(\omega) = \frac{1}{j\omega \bar{r}_i + \bar{s}_i} \tag{37}$$

$$\underline{Q}(\omega) = D(H_i)\,[B]\,\underline{F}(\omega). \tag{38}$$

The matrix [B] has dimensions 2n x n and is obtained from [A] by dropping its first n columns. The Fourier transform of the original coordinate vector $\underline{z}$ can be obtained from (38) as

$$\underline{Z}(\omega) = [A]^T\,D(H_i)\,[B]\,\underline{F}(\omega) \tag{39}$$

Due to the special form of $\underline{Z}$, and the fact that the Fourier transform of $\underline{\dot{x}}$ is $j\omega \underline{X}(\omega)$ where $\underline{X}(\omega)$ is the transform of $\underline{x}$, we can deduce that the first n rows of $[A]^T$ are simply $j\omega$ times the last n rows. Thus the matrix [A] has the form

$$[A] = \left[j\omega\,[B]\;,\;[B] \right]. \tag{40}$$

since the rows of [A] appear in complex conjugate pairs it can be rearranged so it takes the form

$$[A] = \begin{bmatrix} j\omega\,[B] & , [B] \\ -j\omega\,[B^*] & , [B^*] \end{bmatrix} \;;\text{and}\; D(H_i) = \begin{bmatrix} D(H_i) & o \\ o & D(H_i^*) \end{bmatrix}. \tag{41}$$

where [B] is now an n x n matrix and $[B]^*$ is its complex conjugate. The equation (39), which transforms the force $\underline{F}(\omega)$ to the response $\underline{Z}(\omega)$, can thus be written in the expanded form

$$\begin{Bmatrix} j\omega\underline{X} \\ \underline{X} \end{Bmatrix} = \begin{bmatrix} j\omega\,[B]^T & -j\omega\,[B^*]^T \\ [B]^T & [B^*]^T \end{bmatrix} \begin{bmatrix} D(H_i) & o \\ o & D(H_i^*) \end{bmatrix} \begin{bmatrix} [B] \\ [B^*] \end{bmatrix} \underline{F}(\omega). \tag{42}$$

The transfer function matrix relating a force input to a displacement output is denoted as [H] and obtained from (42) as

$$\underline{X} = [H(\omega)]\;\underline{F} \tag{43}$$

where $[H(\omega)] = [B]^T\,D(H_i)\,[B] + [B^*]^T\,D(H_i)\,[B^*].$ (44)

A typical element of $[H(\omega)]$ can be developed in the same manner as described in the case of proportional damping. The primary difference is that the [B] matrix here which corresponds to the [A] matrix in that section has complex elements, whereas [A] had only real elements. Equation (31) then takes the form

$$h_{ij}(\omega) = \sum_{k=1}^{n} \frac{B_k \, b_{ki} b_{kj}}{j\omega - p_k} + \frac{B_k^* \, b_{ki}^* b_{kj}^*}{j\omega - p_k^*} \qquad (45)$$

$$\text{where} \quad p_k = \frac{\bar{s}_k}{-\bar{r}_k} \quad \text{and} \quad B_k = \frac{1}{\bar{r}_k}. \qquad (46)$$

Eauations (31 and 45) are considered the standard forms for the elements of [H] used for Modal Analysis technique.

CLOSURE

This paper develops in detail the theory and assumptions at the base of modal analysis technique. The general form of the transfer functions assumed in modal parameter estimation is given in equations (31) and (45). The term transfer function in this work refers to the ratio of Fourier transforms of output response to input force. This is common terminology in modal analysis, however, the term is often reserved for the ratio of Laplace Transform. All equations required for the determination of modes of vibration directly from vibration measurements are given. Parameter estimation needed for adequate dynamic modelling and design synthesis based on this vibration data is also included.

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