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MOTION OF AN AXISYMMETRIC RIGID BODY WITH VARIABLE INERTIA MOMENTS

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ABSTRACT

The problem of the motion of a rigid body with fixed point is one of the urgent problems of classical mechanics. The peculiarity of this problem is that, despite the important results achieved by outstanding mathematicians during more than the last two centuries, there is still no complete solution. In this paper, an analytical solution of the problem of motion of an axially symmetrical rigid body with variable inertia moments in resistant medium described by a system of nonlinear differential L. Euler equations, involving the method of partial discretization of nonlinear differential equations, built by A.N. Tyurekhodjaev on the basis of the theory of generalized functions [1].

KEY WORDS

Symmetrical rigid body, spherical motion, variable moment of inertia, medium with resistance

NOMENCLATURE

$A(t), B(t), C(t)$	Moments of the body inertia relative to x, y, z axes –, connected with the body.
p, q, r	Projections of a vector of angular velocity of body on these axes.
M_x, M_y, M_z	Moments of external resistance forces relative to x, y, z axes.
$\lambda_1, \lambda_2, \lambda_3$	Arbitrary parameters, depending on the properties of the medium.
$\delta(t)$	Dirac delta function.
$H(t)$	Heaviside function.

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INTRODUCTION

It took more than two hundred years since the publication of the equations of dynamics of a rigid body with fixed point; nevertheless research has not yet been terminated. Great interest in this problem is due to the fact that in the motion of a rigid body with fixed point gyroscopic effects are observed, which became widespread in modern technology, navigation, space technology and many other areas [2]. Topicality of considering the problem of the motion of a rigid body with fixed point is also due to the need to allow for the perturbing gravitational, electrical, magnetic and other forces, to variability of the moment of object inertia and a wide application in practice.

Mathematically, these problems can be reduced to the study of systems of nonlinear differential equations and differential equations with variable coefficients. It is very difficult to obtain analytical solutions for them and it is possible in a relatively small number of cases [3]. Therefore, the construction of analytical solutions for a wide class of such problems is very important.

MOTION OF A SYMMETRIC RIGID BODY AROUND FIXED POINT WITH A VARIABLE INERTIA MOMENT IN A RESISTANT MEDIUM

Consider the equations of motion of a rigid body with fixed point with a variable inertia moment, which is described by a system of nonlinear dynamical equations of L. Euler

$$\begin{cases} \dfrac{dA(t)p}{dt} + [C(t) - B(t)]qr = M_x, \\ \\ \dfrac{dB(t)q}{dt} + [A(t) - C(t)]rp = M_y, \\ \\ \dfrac{dC(t)r}{dt} + [B(t) - A(t)]pq = M_z. \end{cases} \tag{1}$$

Consider the system of differential equations (1) jointly with initial conditions

$$\begin{aligned} t = 0: \quad & p(0) = p_0, \quad q(0) = q_0, \quad r(0) = r_0, \\ & \dot{p}(0) = \dot{p}_0, \quad \dot{q}(0) = \dot{q}_0, \quad \dot{r}(0) = \dot{r}_0. \end{aligned} \tag{2}$$

We do not restrain a generality of the problem, so hereinafter we will consider that:

$$\dot{p}(0) = 0, \quad \dot{q}(0) = 0, \quad \dot{r}(0) = 0. \tag{3}$$

Let the moments of external resistance forces be proportional to the corresponding projections of angular velocity of body

$$M_x = -\lambda_1 p, \quad M_y = -\lambda_2 q, \quad M_z = -\lambda_3 r. \tag{4}$$

Consider a case of symmetrical gyroscope $A(t) = B(t)$. Then the system of nonlinear differential equations (1) will be

$$\begin{cases} \dot{p} + \mu(t) rq + \left[k_1(t) + \dfrac{1}{A(t)} \dfrac{dA(t)}{dt} \right] p = 0, \\ \dot{q} \; - \mu(t) rp + \left[k_2(t) + \dfrac{1}{A(t)} \dfrac{dA(t)}{dt} \right] q = 0, \\ r = r_0 \dfrac{C(0)}{C(t)} e^{-\int_0^t k_3(t) dt}, \end{cases} \tag{5}$$

where

$$\frac{C(t) - A(t)}{A(t)} = \mu(t), \quad \frac{\lambda_1}{A(t)} = k_1(t), \quad \frac{\lambda_2}{A(t)} = k_2(t), \quad \frac{\lambda_3}{C(t)} = k_3(t).$$

From the system (5) for determining $p(t)$ projection of angular velocity of body we have the following differential equation with variable coefficients

$$\begin{aligned} \ddot{p} &+ \left(k_1(t) + k_2(t) + k_3(t) - \frac{\dot{\mu}(t)}{\mu(t)} + \frac{2}{A(t)} \frac{dA(t)}{dt} \right) \dot{p} + \left[\left(\mu(t) r_0 \, \frac{C(0)}{C(t)} e^{-\int_0^t k_3(t) dt} \right)^2 + \right. \\ &+ k_1(t) \left(k_2(t) + k_3(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right) + \dot{k}_1(t) + \frac{1}{A(t)} \frac{d^2 A(t)}{dt^2} + \\ &\left. + \left(k_1(t) - k_3(t) + \frac{\dot{\mu}(t)}{\mu(t)} \right) \frac{1}{A(t)} \frac{dA(t)}{dt} \right] p = 0. \end{aligned} \tag{6}$$

To solve the equation (6), by using the method of partial discretization of nonlinear differential equations [3], we get:

$$\ddot{p}+\left(k_1(t)+k_2(t)+k_3(t)-\frac{\dot{\mu}(t)}{\mu(t)}+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)\dot{p}=$$

$$=-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\left\{\left[\left(\mu(t_i)r_0\,\frac{C(0)}{C(t_i)}e^{-\int_0^{t_i}k_3(t)dt}\right)^2+k_1(t_i)\left(k_2(t_i)+k_3(t_i)-\frac{\dot{\mu}(t_i)}{\mu(t_i)}\right)+\right.\right.$$

$$\left.+\dot{k}_1(t_i)+\frac{1}{A(t_i)}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_i}\right]p(t_i)\delta(t-t_i)-$$

$$-\left[\left(\mu(t_{i+1})r_0\,\frac{C(0)}{C(t_{i+1})}e^{-\int_0^{t_{i+1}}k_3(t)dt}\right)^2+k_1(t_{i+1})\left(k_2(t_{i+1})+k_3(t_{i+1})-\frac{\dot{\mu}(t_{i+1})}{\mu(t_{i+1})}\right)+\right.$$

$$\left.\left.+k_1(t_{i+1})+\frac{1}{A(t_{i+1})}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_{i+1}}\right]p(t_{i+1})\delta(t-t_{i+1})\right\}. \tag{7}$$

The general solution of Eqn. (7) is expressed by:

$$p=C_2+C_1\int\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\times$$

$$\times\left\{\left[\left(\mu(t_i)r_0\,\frac{C(0)}{C(t_i)}e^{-\int_0^{t_i}k_3(t)dt}\right)^2+k_1(t_i)\left(k_2(t_i)+k_3(t_i)-\frac{\dot{\mu}(t_i)}{\mu(t_i)}\right)+\dot{k}_1(t_i)+\frac{1}{A(t_i)}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_i}\right]\times\right.$$

$$\times\frac{p(t_i)}{\mu(t_i)}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_i}}H(t-t_i)\int_{t_i}^{t}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt$$

$$-\left[\left(\mu(t_{i+1})r_0\,\frac{C(0)}{C(t_{i+1})}e^{-\int_0^{t_{i+1}}k_3(t)dt}\right)^2+k_1(t_{i+1})\left(k_2(t_{i+1})+k_3(t_{i+1})-\frac{\dot{\mu}(t_{i+1})}{\mu(t_{i+1})}\right)+\right.$$

$$+\dot{k}_1(t_{i+1})+\frac{1}{A(t_{i+1})}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_{i+1}}\Bigg]\frac{p(t_{i+1})}{\mu(t_{i+1})}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_{i+1}}}\times$$
$$\times H(t-t_{i+1})\int_{t_{i+1}}^{t}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt\Bigg\} \tag{8}$$

where C_1 and C_2 are arbitrary constants of integration.

Using the initial conditions (2), (3), gives

$$p=p_0-\frac{1}{2}\sum_{i=1}^{n}(t_i+t_{i+1})\times$$
$$\times\Bigg\{\Bigg[\Bigg[\left(\mu(t_i)r_0\frac{C(0)}{C(t_i)}e^{-\int_0^{t_i}k_3(t)dt}\right)^2+k_1(t_i)\left(k_2(t_i)+k_3(t_i)-\frac{\dot{\mu}(t_i)}{\mu(t_i)}\right)+\dot{k}_1(t_i)+\frac{1}{A(t_i)}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_i}\Bigg]\times$$
$$\times\frac{p(t_i)}{\mu(t_i)}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_i}}H(t-t_i)\int_{t_i}^{t}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt$$
$$-\Bigg[\left(\mu(t_{i+1})r_0\frac{C(0)}{C(t_{i+1})}e^{-\int_0^{t_{i+1}}k_3(t)dt}\right)^2+k_1(t_{i+1})\left(k_2(t_{i+1})+k_3(t_{i+1})-\frac{\dot{\mu}(t_{i+1})}{\mu(t_{i+1})}\right)+$$
$$+\dot{k}_1(t_{i+1})+\frac{1}{A(t_{i+1})}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_{i+1}}\Bigg]\frac{p(t_{i+1})}{\mu(t_{i+1})}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_{i+1}}}\times$$
$$\times H(t-t_{i+1})\int_{t_{i+1}}^{t}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt\Bigg\} \tag{9}$$

From the latter equation for the first several points of the integral curve, we get the following analytical expressions

$p(t_1)=p_0$,

$$p(t_2) = p_0 - \frac{1}{2}(t_1 + t_2) \times \Bigg\{ \Bigg[\Bigg(\mu(t_1) r_0 \, \frac{C(0)}{C(t_1)} e^{-\int_0^{t_1} k_3(t)dt} \Bigg)^2 +$$

$$+ k_1(t_1)\left(k_2(t_1) + k_3(t_1) - \frac{\dot{\mu}(t_1)}{\mu(t_1)} \right) + \dot{k}_1(t_1) + \frac{1}{A(t_1)}\left[\frac{d^2 A(t)}{dt^2} \right]_{t=t_1} \Bigg] \times$$

$$\times \frac{p(t_1)}{\mu(t_1)} e^{\left[\int \left(k_1(t) + k_2(t) + k_3(t) + \frac{2}{A(t)} \frac{dA(t)}{dt} \right) dt \right]_{t=t_1}} \int_{t_1}^{t_2} \mu(t) e^{-\int \left(k_1(t) + k_2(t) + k_3(t) + \frac{2}{A(t)} \frac{dA(t)}{dt} \right) dt} dt$$

$$p(t_3) = p_0 - \frac{1}{2}(t_1 + t_2) \times \Bigg\{ \Bigg[\Bigg(\mu(t_1) r_0 \, \frac{C(0)}{C(t_1)} e^{-\int_0^{t_1} k_3(t)dt} \Bigg)^2 +$$

$$+ k_1(t_1)\left(k_2(t_1) + k_3(t_1) - \frac{\dot{\mu}(t_1)}{\mu(t_1)} \right) + \dot{k}_1(t_1) + \frac{1}{A(t_1)}\left[\frac{d^2 A(t)}{dt^2} \right]_{t=t_1} \Bigg] \times$$

$$\times \frac{p(t_1)}{\mu(t_1)} e^{\left[\int \left(k_1(t) + k_2(t) + k_3(t) + \frac{2}{A(t)} \frac{dA(t)}{dt} \right) dt \right]_{t=t_1}} \int_{t_1}^{t_3} \mu(t) e^{-\int \left(k_1(t) + k_2(t) + k_3(t) + \frac{2}{A(t)} \frac{dA(t)}{dt} \right) dt} dt -$$

$$- \frac{1}{2}(t_3 - t_1) \times \Bigg\{ \Bigg[\Bigg(\mu(t_2) r_0 \, \frac{C(0)}{C(t_2)} e^{-\int_0^{t_2} k_3(t)dt} \Bigg)^2 +$$

$$+ k_1(t_2)\left(k_2(t_2) + k_3(t_2) - \frac{\dot{\mu}(t_2)}{\mu(t_2)} \right) + \dot{k}_1(t_2) + \frac{1}{A(t_2)}\left[\frac{d^2 A(t)}{dt^2} \right]_{t=t_2} \Bigg] \times$$

$$\times\frac{p(t_2)}{\mu(t_2)}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_2}}\int_{t_2}^{t_3}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt\ ,$$

Using the mathematical induction technique, we construct an analytical expression of the initial function at arbitrary point $t_i\left(i=\overline{1,n}\right)$

$$p(t_i)=p_0-\frac{1}{2}(t_1+t_2)\times\left\{\left[\left[\left(\mu(t_1)r_0\,\frac{C(0)}{C(t_1)}e^{-\int_0^{t_1}k_3(t)dt}\right)^2+\right.\right.\right.$$

$$\left.+k_1(t_1)\left(k_2(t_1)+k_3(t_1)-\frac{\dot{\mu}(t_1)}{\mu(t_1)}\right)+\dot{k}_1(t_1)+\frac{1}{A(t_1)}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_1}\right]\times$$

$$\times\frac{p(t_1)}{\mu(t_1)}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_1}}\int_{t_1}^{t_i}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt-$$

$$-\frac{1}{2}\sum_{j=2}^{i-1}\left(t_{j+1}-t_{j-1}\right)\times\left\{\left[\left[\left(\mu(t_j)r_0\,\frac{C(0)}{C(t_j)}e^{-\int_0^{t_j}k_3(t)dt}\right)^2+\right.\right.\right.$$

$$\left.+k_1(t_j)\left(k_2(t_j)+k_3(t_j)-\frac{\dot{\mu}(t_j)}{\mu(t_j)}\right)+\dot{k}_1(t_j)+\frac{1}{A(t_j)}\left[\frac{d^2A(t)}{dt^2}\right]_{t=t_j}\right]\times \tag{10}$$

$$\times\frac{p(t_j)}{\mu(t_j)}e^{\left[\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt\right]_{t=t_j}}\int_{t_j}^{t_i}\mu(t)e^{-\int\left(k_1(t)+k_2(t)+k_3(t)+\frac{2}{A(t)}\frac{dA(t)}{dt}\right)dt}dt\ ,$$

Substituting obtained expression (10) into (9), gives final expression of the problem in question (6)-(2), (3).

CONCLUSION

A. N. Tyurekhodjaev showed that ordinary and in partial derivatives differential equations with variable coefficients are equivalent to the corresponding nonlinear differential equations, order of which are lower than the order of the initial differential equations.

In this paper, an analytical solution of L. Euler equations of motion, describing the motion in resistant medium of a rigid axisymmetric body with a fixed point with an arbitrary variable moment of inertia using the method of partial discretization of nonlinear differential equations.

REFERENCES

[1] A. N. Tyurekhodjaev, M.A. Bersugir and G.U. Mamatova. "On the Motion of a Ggyroscope in a Resistant Medium", *Proceedings of the International scientific Conference "Modern technology and quality management in education, science and industry: the experience of adaptation and implementation",* Part 1, pp.160-164, Bishkek (2001).

[2] B.V. Bulgakov, "*Applied Theory of Gyroscopes",* Moscow, Publishing House of Moscow University, pp. 258 (1976).

[3] V.N. Koshlyakov, "*Problem of Rigid Body Dynamics and Applied Theory of Gyroscopes",* Moscow, Publishing House "Nauka", pp. 286 (1985).