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MODELING STRESS OF SMART BEAM WITH THERMO-PIEZOELECTRIC EFFECTS

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ABSTRACT

In the present work, stress modeling using finite element technique is proposed to describe the response of laminated composite beam with piezoelectric actuators due to mechanical, electrical, and thermal loads. The assumed field displacements equations are represented by first-order shear deformation theory (FSDT), the Timoshenko beam theory. The equation of motion of the smart beam system is derived using the principle of minimum potential energy. A cubic shape function is used to represent the axial displacement *u*, a quadratic shape function for the transverse displacement w, where the normal rotation ϕ_x, electric potential φ , and thermal temperature θ are represented by a linear shape. A MATLAB code is developed to compute the static and stresses deformations of the structure system due to the thermal loads. The shear correction factor is used to improve the obtained results. The obtained results are compared to the available results of other investigators, good agreement is generally obtained.

KEY WORDS

Finite element method, piezoelectric actuators, thermal stress, Timoshenko beam theory, smart composite structure.

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NOMENCLATURE

Symbol	Definition
A	Beam cross section area
C_{ijkl}	Elastic constants
D_i	Electric displacements
E	Young's modulus
E_1, E_2	Young's modulus in fiber and its transversal direction, respectively
E_k	Electric field (k=1,2,3)
e_{ijk}	Piezoelectric constituents' constants
f_a , f_t	Axial and transversal forces
H	Electric enthalpy
h_c	Convection coefficient
$\lfloor K_{qq} \rfloor$	Mechanical stiffness matrix
$\lfloor K_{\varphi\varphi} \rfloor$	Electric stiffness matrix
$\lfloor K_{q\varphi} \rfloor$	Coupled mechanical - electrical stiffness matrix
$[K_{\theta\theta}]$	Thermal stiffness matrix
$\lfloor K_{q\theta} \rfloor$	Coupled mechanical - thermal stiffness matrix
$\lfloor K_{\varphi\theta} \rfloor$	Coupled electric-thermal stiffness matrix
k_x	Thermal conductivity in x direction
L	Length of beam element
P_i	Pyroelectric constants
$\{Q\}$	Thermal load vector
Q_{ij}	Components of the lamina stiffness matrix
$\ddot{q}$	The second derivative of the nodal displacement
q_x	Heat flux in the x direction
q^*	The rate of energy is generation per unit volume
T_e	Kinetic energy
T	Temperature of the solid /fluid interface
T_∞	Reference temperature
u, v, w	Displacements of any point in the x-, y-, and z directions
$u^\circ, v^\circ, w^\circ$	Reference surface displacements along x-, y-, and z- axes
u_1, u_2, u_3, u_4	Axial displacements at the element nodes
W	Work done external loads
w_1, w_2, w_3	Transversal displacements at the element nodes
γ°_{xy}	In-plane shear strain

γ_{xz}	Transversal shear strain in x-z plane
ε_x, ε_y, ε_z	Linear strains in the x-,y-, and x-directions
ε_x^0 , ε_y^0	Reference surface extensional strains in the x-, and y-directions
ε_{ij}^s	Permittivity constants
κ_x^0, κ_y^0	Reference surface curvatures in the x-, and y-directions
ξ_i, ζ_i	Axial and transversal displacements shape functions
Π	Total potential energy
ρ	Mass density of structure material
σ_{xx}	Normal stress in the x-direction
σ_{xz}	Shear stress in the x-z plane
σ	surface charge density
ϕ_x	Angle of rotation
α_i	Thermal expansion coefficients
φ	Electric potential
ψ_i	Rotation displacement shape functions
θ_1 , θ_2	Temperature degrees of freedom

Abbreviations

HSDT	Higher-order shear deformation Theory
SSDT	Second-order shear deformation Theory
FSDT	First-order shear deformation Theory
CBT	Classical beam Theory

INTRODUCTION

Several researchers are interested to solve smart beams have piezoelectric actuators with thermal effect using different theories. Nowacki [1] presented a uniqueness theorem for the solutions of the different equations of thermo-piezoelectricity, on the basis of the energy balance. The generalized Hamilton principle and the theorem of reciprocity of work are also deduced in this study.

Tzou and Ye [2] presented the piezothermoelastic effects of piezoelectric laminates with distributed piezoelectric sensor/ actuator subjected to a steady–state temperature field. They defined the piezothermoelastic constitutive equations, followed by three energy functional for the displacement, electric, and temperature fields, respectively. The distributed sensing and control equations in temperature fields are also defined. They formulated 3D piezothermoelastic thin hexahedron finite element with three internal degrees of freedom using a variational formulation which includes thermal, electrical, and mechanical energies. Thermal influences on the sensing and control of piezoelectric PZT/steel laminates are also investigated.

Tzou and Howard [3] presented the piezothermoelastic characteristics of the piezoelectric shell continua and the application of the theory to active structure in sensing and control. The study showed that the piezothermoelastic equations in three principle directions include thermal induced loads, as well as, conventional electric and mechanical loads. Also the electric membrane forces and moments induced by the converse effect can be used to control the thermal and mechanical loads. They proposed simplification procedure, based on the Lame' parameters and radii of curvatures and applied the theory to piezoelectric cylindrical shell, piezoelectric ring, and piezoelectric beam.

Lee and Saravanos [4] developed a discrete layer model to incorporate thermal effects to account for the complete coupled mechanical, electrical, and thermal response of the piezoelectric composite beams. Their representation leads to an inherent capability to model both the sensory and active responses of piezoelectric composite beams in thermal environments. They gave additional numerical studies demonstrated the capabilities of their model to predict the thermal deformation of composite beams, the active compensation of these thermal deformation, the corresponding sensory response, and the resultant stress state in the piezoelectric structure.

Lee and Saravanos [5] proposed analytical formulations for thermo piezoelectric composite materials accounted for thermal effects from temperature dependant material properties. The thermal effects are represented, at the material levels, through the thermo piezoelectric constitutive equations incorporated into a layer wise laminate theory. A corresponding finite element equation was derived and implemented for beams and plates to model the active and sensory response of piezoelectric composite laminates.

Raja et al. [6] proposed finite element formulation of a laminated beam with embedded piezoelectric material as distributed actuators/sensors. They derived a two nodded 3-D beam element using first shear deformation theory to model the direct and coupled effects. A feed back is employed to actively control the first three modes of a cantilever PZT/steel/PZT beam. The model included contributions from elastic, thermo elastic, piezoelectric, dielectric, pyroelectric and heat components. Both thermal strain effect and direct pyroelectric effect generated sensory voltages, however, the thermal strain effect is of greater significance. They employed eigen structure assignment technique using output feedback. The first three dynamic modes are controlled by placing the desired eigen values exactly. And the tip motion of the cantilever smart beam is significantly reduced by shaping the eigenvectors of the closed systems.

Lee [7] proposed comprehensive study for the behavior of composite laminate contains piezo-electric elements. The study is emphasis on the coupled response at the material level through the thermo-piezoelectric constitutive equations and to introduce the displacements, electric potential, and temperature as state variables in the analysis. The mechanics incorporate a layer wise laminate theory for more accurate analysis especially for thick laminates. The thermal effect, pyroelectric effects, and temperature dependent material properties are also accounted. The corresponding finite element formulations are developed for beam, plate, and shell elements for a static and dynamic analysis of the structures. The obtained results from this model are compared with the published experimental data and analytical models and found reasonable.

Shang et al. [8] proposed finite element techniques for three-dimensional coupled thermo-electro-mechanical static analysis. The actual thermo-piezoelectric responses subjected to thermal loadings can be determined by adopting a procedure TPESAP. The detailed implementation is presented with emphasis on the integration with software ABAQUS. A three-dimensional (3D) formulation of thermo-piezoelectric problems is presented for general purpose, making use of the weak form of thermo-electro-mechanical equilibrium.

Bansal and Ramaswamy [9] proposed finite element model to study the dynamic as well as the static thermal response of laminated composite containing distributed piezoelectric layers. They studied both the sensory and active responses on piezoelectric composite beam and plate structure. They concluded that both the thermal and pyroelectric effects are important and need to be considered in the precision distribution control of intelligent structures. Also these thermal effects will improve the performance characteristics of the system.

Görnandt and Gabbert [10] presented weak form of the fully coupled thermopiezomechanical field equations, including linearized constitutive equations, developed on the basis of the balance laws of the mechanics and electro dynamics and the entropy production inequality. The obtained results from their model show that, upon a finite element discrete in space, the governing semi-discrete system of differential equations is unsymmetrical. The temperature is coupled with the mechanical displacement and electric potential only via the first time derivatives. They found that for static case these equations can be solved separately, and consequently, also for dynamic cases a separate solution scheme is proposed where the solutions for each time step are calculated iteratively. They concluded that temperature may exert a considerable on the behavior of the controlled smart structure.

Benjeddou and Andrianarison [11] proposed piezoelectric mixed variational theorem (PMVT), by adding the transverse thermal field-temperature increment relations a constraint via a Lagrange multiplier. Where the Lagrange multiplier shown that the transverse (normal) heat flux which continuity can be fulfilled in a natural way as the case for the transverse structure stress and transverse electric field in (PMVT). Therefore, they presented a thermo-piezo-electric mixed variational theorem (TMVT) which is suited for implementing analytical or closed form and numerical, such as finite element, solutions for thermo piezoelectric multilayered smart composites. They developed a mixed thermo piezo-electric constitutive equations to be used in conjunction with the TMVT and a guidelines for their numerical implementation are also given.

Jonnalagadda et al. [12] proposed finite element model to compute the response of composite plates with attached piezoelectric PVDF type subjected to mechanical, electrical, and thermal loads based on the first order shear deformation theory. They found that moderately thick piezo-thermo-elastic composite are sensitive to shear deformation and the influence of this shear deformation diminishes with increase plate thickness and aspect ratio.

Senthil and Batra [13] proposed generalized plane strain thermo piezoelectric deformations of laminated thick plates using the Eshelby–Stroh formalism where the analytical solution is in terms of an infinite series. The formulation admits different thermal, electrical, and mechanical boundary conditions at the edges of each lamina, and it is applicable to thick and thin laminated plates. The computed results prove the versatility of the proposed technique for obtaining accurate stresses for thick hybrid multilayered plates subjected to various thermal, electrical, and mechanical loads.

Wang and Noda [14] proposed finite element model of smart functionally graded piezoelectric structure with thermal effect. The results reveal that both the stress discontinuity and thermal deformation of the structure can be controlled. They introduced a functionally graded between the PZT actuator layer and the metal beam layer, they concluded that both stress discontinuity and the edge local stresses can be essentially reduced.

Lepage [15] developed a computational framework to determine the thermoelastic quality factor and the modeling of uncertainties. He derived a thermopiezoelectric finite element formulation to carry out model analyses of MEMS. The perturbation stochastic finite element method (SFEM) is used to determine the mean and variance of the thermo-elastic quality factor and is compared to direct Monte-Carlo simulations.

Ashida and Noda [16] presented a model of the piezoelectric based intelligent solid state actuator which could control the elastic displacement distribution, adapting to the transient temperature change of the surrounding medium. The model of the actuator consists of isotropic structural layer onto which two piezoceramic layers are perfectly bonded. By analyzing the intelligent problem of transient piezothermoelastic field in the composite plate, it was shown how to sense the unknown heating temperature and to control the elastic displacement distribution of the structure. Numerical simulation demonstrated that the elastic displacement distribution of the structural layer induced by the unknown transient heating temperature could be controlled by applying the determined electric potential distribution.

Piening [17] proposed 2D models which was suited to predict the strength behavior of homogeneous metallic structures. He concluded that the as soon as the elastic body system becomes inhomogeneous, has a complex shape and or an isotropic, the solution was at least questionable. Transverse stresses can not be calculated and the effect of changes in the layer stacking sequence can not be estimated. But the results of static and fatigue test showed a strong dependence of the failure behavior on the stacking order. Also the action of the active layers, the placement of active layers within the cross-section has a decisive effect on the long term efficiency of the active laminate, and must be taken into consideration in the early design phase.

The present work proposed a stresses modeling of the smart piezoelectric laminated beams with thermo-piezoelectric effects using a finite element technique. The model is based on the FSDT, Timoshenko theory, which taking into consideration the shear effect and used shear correction factor. The structure system response due to mechanical, electrical, and thermal loads are obtained and compared with other investigator and found reasonable.

THEORETICAL FORMULATION

The displacements field equations for Timoshenko first-order shear deformation theory (FSDT) at any point through the thickness of a beam can be expressed as [18]:

$$\begin{gathered} u(x,z) = u_0(x) - z\phi_x(x) \\ v(x,z) = 0 \\ w(x,z) = w_\circ(x) \end{gathered} \tag{1}$$

where u, v and w are the displacements field equations along the x, y and z coordinates, respectively, u_0 and w_0 denote the displacements of a point $(x, y, 0)$ at the mid plane, and $\phi_x(x)$ is the rotation angle of the cross-section.

According to the assumptions of the first order Timoshenko beam theory the only non-zero stress and strain components are $\sigma_{xx}, \sigma_{xz}, \varepsilon_{xx}, \gamma_{xz}$ $(\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = \gamma_{yz} = 0)$ [18]. Thus the strain-displacement relationships are obtained by differentiating the assumed displacements field equations, Eqn. (1), as follows:

$$\varepsilon_{xx}(x,y,z) \equiv \frac{\partial u(x,y,z)}{\partial x} = \frac{\partial u_\circ(x,z)}{\partial x} - z\,\frac{\partial \phi_x(x,z)}{\partial x} = \varepsilon^\circ_{xx} + z\,\kappa^\circ_{xx} \tag{2 a}$$

$$\gamma_{xz}(x,y,z) \equiv \frac{\partial u(x,y,z)}{\partial z} + \frac{\partial w(x,y,z)}{\partial x} = \phi_x + \frac{\partial w_0}{\partial x} = \gamma^0_{xz} \tag{2 b}$$

where, ε°_{xx} is the reference surface extensional strain in the x-direction, γ°_{xz} is the in-plane shear strain, and κ°_{xx} is the reference surface curvature in the x-direction. The strains at any point through the thickness of the beam can be written in matrix form as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^0_{xx} \\ \gamma^0_{xz} \end{Bmatrix} + z \begin{Bmatrix} \kappa^0_{xx} \\ \kappa^0_{xz} \end{Bmatrix}, \tag{3 a}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ -\phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} -\dfrac{\partial \phi_x}{\partial x} \\ 0 \end{Bmatrix}. \tag{3 b}$$

Thermopiezoelectric Constitutive Relations

For the elastic system with piezoelectric material, the total potential energy H stored in a lamina comprises the various components of the elastic strain energy, piezoelectric material energy and electrical energy, which given by [15, 19, 20, and 21]:

$$H\left(\varepsilon_{ij},E_i,\theta\right)=\frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-e_{ijk}E_i\varepsilon_{jk}-\frac{1}{2}\varepsilon_{ij}^{s}E_iE_j-\beta_{ij}\varepsilon_{ij}\theta-P_iE_i\theta-\frac{1}{2}\lambda_T\theta^2 \tag{4}$$

where; $i,j=1,...,6$, and $k,l=1,...3$, c_{ijkl}, e_{ijk}, and ε_{ij}^{s} are the elastic, piezoelectric, and permittivity constants, respectively, β_{ij} is the thermal mechanical constants ($\beta_{ij}=c_{ijkl}\alpha_{kl}$) and α_{kl} is the thermal expansion coefficient, ε_{ij} is the mechanical strain components, P_i is the pyroelectric constant, and θ is the temperature rise from the initial temperature T_∞ and the temperature T of the solid /fluid interface ($\theta=T-T_\infty$). The parameter λ_T is defined as $\lambda_T=\frac{C_E}{T_\infty}$, where C_E is the heat capacity. The constitutive relations of the thermal-piezoelectricity are obtained by differentiating the function H with respect to the strain, electric field and the temperature rise components as follows:

$$\sigma_{ij}=\frac{\partial H}{\partial\varepsilon_{ij}}=c_{ijkl}\varepsilon_{kl}-e_{ijk}E_k-\beta_{ij}\theta \tag{5}$$

$$D_i=-\frac{\partial H}{\partial E_i}=e_{ijk}\varepsilon_{jk}+\varepsilon_{ij}^{s}E_j+P_i\theta \tag{6}$$

$$S=-\frac{\partial H}{\partial\theta}=\beta_{ij}\varepsilon_{ij}+P_iE_i+\lambda_T\theta \tag{7}$$

where S is the entropy density.

The constitutive relations with plane stress approximation for k^{th} can be written as follows [7]:

$$\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{Bmatrix}_k=\begin{bmatrix}Q_{11}&Q_{12}&0&0&0\\Q_{12}&Q_{22}&0&0&0\\0&0&Q_{44}&0&0\\0&0&0&Q_{55}&0\\0&0&0&0&Q_{66}\end{bmatrix}_k\begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{23}\\\varepsilon_{13}\\\varepsilon_{12}\end{Bmatrix}_k-\begin{bmatrix}0&0&e'_{31}\\0&0&e'_{32}\\0&e'_{24}&0\\e'_{15}&0&0\\0&0&0\end{bmatrix}_k\begin{Bmatrix}E_1\\E_2\\E_3\end{Bmatrix}_k-\begin{Bmatrix}\beta_1\\\beta_2\\0\\0\\0\end{Bmatrix}_k\theta \tag{8}$$

$$\begin{Bmatrix}D_1\\D_2\\D_3\end{Bmatrix}_k=\begin{bmatrix}0&0&0&e'_{15}&0\\0&0&e'_{24}&0&0\\e'_{31}&e'_{32}&0&0&0\end{bmatrix}_k\begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{23}\\\varepsilon_{13}\\\varepsilon_{12}\end{Bmatrix}_k+\begin{bmatrix}\varepsilon'^{s}_{11}&0&0\\0&\varepsilon'^{s}_{22}&0\\0&0&\varepsilon'^{s}_{33}\end{bmatrix}_k\begin{Bmatrix}E_1\\E_2\\E_3\end{Bmatrix}_k+\begin{Bmatrix}P_1\\P_2\\P_3\end{Bmatrix}\theta \tag{9}$$

where;
$$e'_{31}=e_{31}-\frac{c_{13}}{c_{33}}e_{33};\quad e'_{32}=e_{32}-\frac{c_{23}}{c_{33}}e_{33};\quad e'_{24}=e_{24};\quad e'_{15}=e_{15};$$
$$\varepsilon'^{s}_{11}=\varepsilon^{s}_{11};\quad \varepsilon'^{s}_{22}=\varepsilon_{22};\quad \varepsilon'^{s}_{33}=\varepsilon_{33}+\frac{e_{33}^2}{c_{33}} \tag{10}$$

The elastic coefficients c_{ij} are given by Ref. [22], and the reduced stiffness coefficients Q_{ij} related to the engineering constants are defined for two cases as follows [23]:

Case I: Isotropic layer

$$Q_{11}^k = Q_{22}^k = \frac{E}{1-\nu^2}. \qquad Q_{12}^k = \frac{\nu E}{1-\nu^2} \qquad Q_{44}^k = Q_{55}^k = Q_{66}^k = G \tag{11 a}$$

where E , G , and ν are the isotropic material properties.

Case II: Orthotropic layer

$$Q_{11}^k = \frac{E_1^k}{1-\upsilon_{12}^k \upsilon_{21}^k} \qquad Q_{12}^k = \frac{\upsilon_{12}^k E_2^k}{1-\upsilon_{12}^k \upsilon_{21}^k} \qquad Q_{22}^k = \frac{E_2^k}{1-\upsilon_{12}^k \upsilon_{21}^k}$$
$$Q_{44}^k = G_{23} \qquad Q_{55}^k = G_{13}^k \qquad Q_{66}^k = G_{12}^k \tag{11 b}$$

Thus the transformed relations from the material principle axes 1, 2, and 3 to the geometrical axes x, y, and z can be written as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ \bar{e}_{14} & \bar{e}_{24} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \\ 0 & 0 & \bar{e}_{36} \end{bmatrix}_k \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}_k - \begin{Bmatrix} \bar{\beta}_{xx} \\ \bar{\beta}_{yy} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \theta \tag{12}$$

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix}_k = \begin{bmatrix} 0 & 0 & \bar{e}_{14} & \bar{e}_{15} & 0 \\ 0 & 0 & \bar{e}_{24} & \bar{e}_{25} & 0 \\ \bar{e}_{31} & \bar{e}_{32} & 0 & 0 & \bar{e}_{36} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k + \begin{bmatrix} \bar{\varepsilon}_{xx}^s & \bar{\varepsilon}_{xy}^s & 0 \\ \bar{\varepsilon}_{xy}^s & \bar{\varepsilon}_{yy}^s & 0 \\ 0 & 0 & \bar{\varepsilon}_{zz}^s \end{bmatrix}_k \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}_k + \begin{Bmatrix} \bar{P}_x \\ \bar{P}_y \\ \bar{P}_z \end{Bmatrix} \theta \tag{13}$$

where $\bar{Q}_{ij}$ and $\bar{e}_{ij}$ are the transformed reduced stiffness coefficients, and piezoelectric modules, respectively. And ($\bar{\beta}_{xx}$, and $\bar{\beta}_{yy}$), ($\bar{P}_x, \bar{P}_y$, and $\bar{P}_z$), are the transformed thermal expansions components, and pyroelectric constants, respectively.

In the proposed model the following assumptions are used: (1) The width in y direction is stress free and the plane stress assumption is used. Therefore, it is possible to set $\sigma_{yy} = \sigma_{yz} = \sigma_{xy} = \gamma_{yz} = \gamma_{xy} = 0$, and $\varepsilon_{yy} \neq 0$. (2) The polarization axis z is aligned with the thickness direction of the beam, thus only D_z in Eqn.(13) is taken into consideration. (3) By introducing E_z applied across the actuator thickness and the other components of the electric fields are zeros. (4) The coefficient e_{15} and ε_{11}^s

are neglected. Therefore the constitutive relations Eqn. (12) and Eqn. (13) are reduced to:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix}_k = \begin{bmatrix} \tilde{Q}_{11} & 0 & 0 \\ 0 & k_s \tilde{Q}_{55} & 0 \\ \tilde{e}_{31} & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \\ 0 \end{Bmatrix}_k - \begin{bmatrix} \tilde{e}_{31} \\ 0 \\ -\tilde{\varepsilon}_{zz} \end{bmatrix}_k E_z - \begin{Bmatrix} \overline{\beta}_{xx} \\ 0 \\ \overline{P}_z \end{Bmatrix}_k \theta \qquad (14)$$

where; k_s is the shear correction factor, $\overline{Q}_{ij}$ and $\overline{e}_{ij}$ are the transformed reduced stiffness coefficients, and the transformed piezoelectric modules, respectively. $\overline{\beta}_{xx}$, $\overline{P}_z$ are the transformed thermal expansions components and pyroelectric constants, respectively [23]. The other coefficients in Eqn. (14) are given by:

Case I: Isotropic layer

$$\tilde{Q}_{11} = E \qquad \tilde{Q}_{55} = G \qquad \text{, and} \qquad \overline{Q}_{ij} = Q_{ij} \qquad (15)\text{ a}$$

Case II: Orthotropic layer

$$\tilde{Q}_{11} = \overline{Q}_{11} - \frac{\overline{Q}_{12}\overline{Q}_{12}}{\overline{Q}_{22}} \qquad \tilde{Q}_{55} = \overline{Q}_{55} - \frac{\overline{e}_{25}}{\overline{e}_{24}}\overline{Q}_{45} \qquad (15)\text{ b}$$

And the piezoelectric coefficients are given by:

$$\tilde{\varepsilon}^s_{zz} = \overline{\varepsilon}^s_{zz} + \frac{\overline{e}_{32}\overline{e}_{32}}{\overline{Q}_{22}} \qquad \tilde{e}_{31} = \overline{e}_{31} - \overline{e}_{32}\frac{\overline{Q}_{12}}{\overline{Q}_{22}} \qquad \tilde{e}_{15} = \overline{e}_{15} - \frac{\overline{e}_{25}\overline{e}_{14}}{\overline{e}_{24}} \qquad (15)\text{ c}$$

The electric field components are related to the electrostatic potential φ by the equation [24]:

$$E_k = -\phi,_k \qquad (16)$$

Energy Formulation

The internal strain energy $\hat{U}$ is represented by [25]:

$$\hat{U} = \frac{1}{2}\int_v \varepsilon_{kl}\sigma_{ij} dv \qquad (17)$$

By substituting Eqn. (5) into (17) one can obtain:

$$\widehat{U} = \frac{1}{2}\int_v \left[c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{ijk}E_i\varepsilon_{jk} - \beta_{ij}\varepsilon_{ij}\theta \right] dv \qquad (18)$$

The electric potential energy is given by:

$$U_e = \frac{1}{2}\int_v E_k D_i dv \qquad (19)$$

By substituting Eqn. (6) into (19) one can obtain:

$$U_e = \frac{1}{2}\int_v \left[e_{ijk} E_i \varepsilon_{jk} + \varepsilon^s_{ij} E_i E_j + P_i E_i \theta \right] dv \tag{20}$$

The internal strain energy for the structure system U is the sum of internal strain energy $\hat{U}$, equation (18) and the electric potential energy U_e, equation (20) such as:

$$U = \hat{U} - U_e \tag{21}$$

$$U = \frac{1}{2}\int_v \left[c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - 2e_{ijk} E_i \varepsilon_{jk} - \beta_{kl}\varepsilon_{kl}\theta - \varepsilon^s_{ij} E_i E_j - P_i E_i \theta \right] dv \tag{22}$$

For the proposed beams the total strain energy is given as:

$$U = \frac{1}{2}\int_v \left[\tilde{Q}_{11}\varepsilon^2_{xx} + k_s \tilde{Q}_{55}\gamma^2_{xz} - 2\tilde{e}_{31} E_z \varepsilon_{xx} - \tilde{\varepsilon}^s_{zz} E^2_z - \bar{\beta}_{xx}\varepsilon_{xx}\theta + \bar{P}_z E_z \theta \right] dv \tag{23}$$

By substituting Eqn. (2) into Eqn. (23) the strain energy is expressed by:

$$U = \frac{1}{2}\int_v \begin{bmatrix} \tilde{Q}_{11}\left(\frac{\partial u_0}{\partial x} - z\frac{\partial \phi_x}{\partial x}\right)^2 + k_s\tilde{Q}_{55}\left(-\phi_x + \frac{\partial w_0}{\partial x}\right)^2 - 2\tilde{e}_{31}\left(-\frac{\partial \varphi}{\partial z}\right)\left(\frac{\partial u_0}{\partial x} - z\frac{\partial \phi_x}{\partial x}\right) \\ -\tilde{\varepsilon}^s_{33}\left(-\frac{\partial \varphi}{\partial z}\right)^2 - \bar{\beta}_{xx}\left(\frac{\partial u_0}{\partial x} - z\frac{\partial \phi_x}{\partial x}\right)\theta + \bar{P}_z\left(-\frac{\partial \varphi}{\partial z}\right)\theta \end{bmatrix} dv \tag{24}$$

The mass matrix can be obtained using the kinetic energy equations such as:

$$T_e = \frac{1}{2}\int_v \rho \left[\dot{u}^2 + \dot{w}^2 \right] dv \tag{25}$$

where, ρ is the mass density of the material.

The work done by the external mechanical, electrical, and thermal loads are expressed as:

$$W_{Total} = W_{Mech} + W_{Electr} + W_{Therm} \tag{26}$$

The first two components in the right hand side of Eqn. (26) are expressed by [29]:

$$W_{Mech} = \int_0^L f_a u dx + \int_0^L f_t w dx + P_i w_i \tag{27}$$

where; f_a, and f_t are the transversal and axial forces along a surface with length L, respectively. P_i, is the concentrated force at point i and w_i is the corresponding generalized displacement.

and

$$W_{Electr} = -\int_{S_1} \sigma\varphi dS_1 \tag{28}$$

where σ (C/m^2) is the surface charge density on the actuator surface, and φ is the electric potential at the piezoelectric surface area S_1 [26].

Heat Equations

The Fourier heat conduction equation in one dimensional can be expressed as [27]:

$$q_x = -k_{xx}\frac{dT}{dx} \tag{29}$$

where q_x is the heat flux in x direction and k_{xx} is the thermal conductivity of the material and T is the temperature. As an analogous to the strain energy function the expression for the heat conduction is expressed as follows:

$$U_c = \frac{1}{2}\int_v \left[k_{xx}\frac{dT}{dx}\right]^2 dv \tag{30}$$

Thus the third component of the right hand side of Eqn. (26) is expressed as [27, 28]:

$$W_{Therm} = f_Q + f_q + f_h \tag{31}$$

where f_Q is the heat source positive, sink negative, it is the same form as the body-force term, f_q is a heat flux, positive into the surface, and f_h is the heat transfer or convection which is similar to surface traction or distributed loading in the stress analysis problem. Thus;

$$W_{therm} = -\int_v \bar{Q}T dv - \int_{S_2} q^* T dS + \frac{1}{2}\int_{S_3} h_c (T - T_\infty)^2 dS \tag{32}$$

where $\bar{Q}$ is the internal heat source (heat generated per unit time per unit volume), S_2 and S_3 are separate surface areas, q^* heat flux (q^* is positive into the surface). The term $h_c(T - T_\infty)$ is the heat flow by convection heat transfer, and h_c is the convection heat transfer coefficient. The values of q^* and h_c can not specify on the same surface because they cannot occur simultaneously on the same place. Thus the total work done can be expressed as [15, 20]:

$$W_{Total} = \int_0^L f_a u dx + \int_0^L f_t w dx + P_i w_i - \int_{S_1} \sigma\varphi dS_1 - \int_v \bar{Q}T dv - \int_{S_2} q^* T dS + \frac{1}{2}\int_{S_3} h_c (T - T_\infty)^2 dS \tag{33}$$

FINITE ELEMENT FORMULATION

A five nodes beam element with thirteen degrees of freedom is shown in Figure 1.The element has nine mechanical degrees of freedom u, w, and ϕ_x which represent the axial, transverse, and rotational at the node, u, w, and ϕ_x, respectively.

Two electric degrees of freedom φ in addition to two thermal degrees of freedom θ as shown in Figure 1 [29].

The axial displacement can be expressed in the nodal displacement as follows:

$$u(x) = u_1\xi_1 + u_2\xi_2 + u_3\xi_3 + u_4\xi_4 = \sum_{j=1}^{4} u_j \xi_j \tag{34}$$

where the cubic shape functions ξ_j are found to be [30]:

$$\begin{aligned}
\xi_1 &= -\frac{9}{2}\left(\frac{x}{L}\right)^3 + 9\left(\frac{x}{L}\right)^2 - \frac{11}{2}\left(\frac{x}{L}\right) + 1 \qquad & \xi_2 &= \frac{27}{2}\left(\frac{x}{L}\right)^3 - \frac{45}{2}\left(\frac{x}{L}\right)^2 + 9\left(\frac{x}{L}\right) \\
\xi_3 &= \frac{-27}{2}\left(\frac{x}{L}\right)^3 + 18\left(\frac{x}{L}\right)^2 - \frac{9}{2}\left(\frac{x}{L}\right) \qquad & \xi_4 &= \frac{9}{2}\left(\frac{x}{L}\right)^3 - \frac{9}{2}\left(\frac{x}{L}\right)^2 + \frac{x}{L}
\end{aligned} \tag{35}$$

The transversal displacement w can be expressed in terms of the nodal displacement as [30]:

$$w(x) = w_1\zeta_1 + w_2\zeta_2 + w_3\zeta_3 = \sum_{j=1}^{3} w_j \zeta_j \tag{36}$$

where, the quadratic interpolation shape functions are given by:

$$\zeta_1 = 1 - 3\left(\frac{x}{L}\right) + 2\left(\frac{x}{L}\right)^2 \quad \zeta_2 = 4\left(\frac{x}{L}\right) - 4\left(\frac{x}{L}\right)^2 \quad \zeta_3 = -\left(\frac{x}{L}\right) + 2\left(\frac{x}{L}\right)^2 \tag{37}$$

The rotation angle ϕ_x is expressed as:

$$\phi_x = \phi_1\psi_1 + \phi_2\psi_1 = \sum_{j=1}^{2} \phi_j \psi_j \tag{38}$$

where the Linear interpolation shape functions ψ_j have the form:

$$\psi_j = 1 - \frac{x}{L} \text{ and } \psi_j = \frac{x}{L} \tag{39}$$

In the proposed model, the electric potential is considered as function of the thickness and the length of the beam [31]. In case of the electric potential is function of the length, it can be represented by:

$$\varphi(x) = \varphi_1\hat{\varsigma}_1 + \varphi_2\hat{\varsigma}_2 = \sum_{j=1}^{2} \varphi_j \hat{\varsigma}_j \tag{40}$$

where; $$\hat{\varsigma}_1 = 1 - \frac{x}{L}, \text{ and } \hat{\varsigma}_2 = \frac{x}{L} \tag{41}$$

And in case the electric potential is function of the thickness of the beam, it can be given as:

$$\varphi(z)=\varphi_1\breve{\varsigma}_1+\varphi_2\breve{\varsigma}_2=\sum_{j=1}^{2}\varphi_j\breve{\varsigma}_j \tag{42}$$

where;
$$\breve{\varsigma}_1=\frac{1}{2}+\frac{z}{h},\qquad \breve{\varsigma}_2=\frac{1}{2}-\frac{z}{h} \tag{43}$$

Thus by the product of equations (41) and (43) and impose the homogenous boundary condition on the bottom surface to eliminate the rigid body modes. Thus the electric potential can be written as:

$$\varphi(x,0,z)=\varphi_1\varsigma_1+\varphi_2\varsigma_2=\sum_{j=1}^{2}\varphi_j\varsigma_j \tag{44}$$

And the shape functions are finally takes the form:

$$\varsigma_1=\left(\frac{1}{2}+\frac{z}{h}\right)\left(1-\frac{x}{L}\right);\ \varsigma_2=\left(\frac{1}{2}+\frac{z}{h}\right)\left(\frac{x}{L}\right) \tag{45}$$

By assuming a linear temperature variation through the element, thus:

$$\theta(x)=\theta_1\eta_1+\theta_2\eta_2=\sum_{i=1}^{2}\theta_1\eta_i \tag{46}$$

where the Linear interpolation shape functions η_i for the temperature distribution have the form:

$$\eta_1=1-\frac{x}{L}\qquad \text{, and}\qquad \eta_2=\frac{x}{L} \tag{47}$$

Variation Formulation

Using the principle of the minimum potential energy by equating the first variation of the total minimum potential energy Π to zero results in:

$$\delta\Pi=\delta(U-W)=0 \tag{48}$$

The first variation of the strain energy equation number (24) takes the forms:

$$\delta U=\int\left[\begin{array}{l}\left[\begin{array}{l}\tilde{Q}_{11}\left[\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)-z\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_x}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)+z^2\left(\frac{\partial\phi_x}{\partial x}\delta\frac{\partial\phi_x}{\partial x}\right)\right]\\ +k_s\tilde{Q}_{55}\left[(\phi_x\delta\phi_x)-\left(\phi_x\delta\frac{\partial w_0}{\partial x}+\frac{\partial w_0}{\partial x}\delta\phi_x\right)+\left(\frac{\partial w_0}{\partial x}\delta\frac{\partial w_0}{\partial x}\right)\right]\end{array}\right]+\\ \tilde{e}_{31}\left[\left(\frac{\partial\varphi}{\partial z}\delta\frac{\partial u_0}{\partial x}+\frac{\partial u_0}{\partial x}\delta\frac{\partial\varphi}{\partial z}\right)-z\left(\frac{\partial\varphi}{\partial z}\delta\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_x}{\partial x}\delta\frac{\partial\varphi}{\partial z}\right)\right]-\tilde{\varepsilon}^s_{33}\left[\left(\frac{\partial\varphi}{\partial z}\delta\frac{\partial\varphi}{\partial z}\right)\right]\\ -\frac{1}{2}\bar{\beta}_{xx}\left[\left(\frac{\partial u_0}{\partial x}\delta\theta+\theta\delta\frac{\partial u_0}{\partial x}\right)-z\left(\frac{\partial\phi_x}{\partial x}\delta\theta+\theta\delta\frac{\partial\phi_x}{\partial x}\right)\right]\\ -\frac{1}{2}\bar{P}_z\left(\frac{\partial\varphi}{\partial z}\delta\theta+\theta\delta\frac{\partial\varphi}{\partial z}\right)\end{array}\right]dv \tag{49}$$

Thus the elements of the stiffness matrix can be written as:

$$K_{11} = A_{11}\int_A \left(\frac{\partial u_o^T}{\partial x}\delta\frac{\partial u_0}{\delta x} \right) dA \qquad K_{13} = -B_{11}\int_A \left(\frac{\partial u_0^T}{\partial x}\delta\frac{\partial \phi_x}{\partial x} \right) dA$$

$$K_{14} = \bar{e}_{31}\int_v z\left(\frac{\partial u_0^T}{\partial x}\delta\frac{\partial \varphi}{\partial z} \right) dv \qquad K_{15} = -\frac{1}{2}\bar{\beta}_{xx}\int_v \left(\frac{\partial u_0^T}{\partial x}\delta\theta \right) dv$$

$$K_{22} = k_s A_{55}\int_A \left(\frac{\partial w_0^T}{\partial x}\delta\frac{\partial w_0}{\partial x} \right) dA \qquad K_{23} = -k_s A_{55}\int_A \left(\delta\frac{\partial w_0^T}{\partial x}\phi_x \right) dA$$

$$K_{24} = \tilde{e}_{31}\int_v \left(\frac{\partial w_0^T}{\partial x}\delta\frac{\partial \varphi}{\partial z} \right) dv$$

$$K_{31} = -B_{11}\int_A \left(\frac{\partial \phi_x^T}{\partial x}\delta\frac{\partial u_0}{\partial x} \right) dA \qquad K_{32} = -k_s A_{55}\int_A \left(\phi_x^T \delta\frac{\partial w_0}{\partial x} \right) dA$$

$$K_{33} = \int_A \left[D_{11}\left(\frac{\partial \phi_x^T}{\partial x}\delta\frac{\partial \phi_x}{\partial x} \right) + k_s A_{55}\left(\phi_x^T\delta\phi_x \right) \right] dA \qquad K_{34} = -\tilde{e}_{31}\int_v \left[z\left(\frac{\partial \phi_x^T}{\partial x}\delta\frac{\partial \varphi}{\partial z} \right) \right] dv \tag{50}$$

$$K_{35} = -\frac{1}{2}\bar{\beta}_{xx}\int_v \left[\left(\frac{\partial \phi_x}{\partial x}\delta\theta \right) \right] dv$$

$$K_{41} = \tilde{e}_{31}\int_V z\left(\frac{\partial \varphi^T}{\partial z}\delta\frac{\partial u_0}{\partial x} \right) dv \qquad K_{42} = \tilde{e}_{31}\int_v \left(\frac{\partial w_0}{\partial x}\delta\frac{\partial \phi}{\partial z} \right) dv$$

$$K_{43} = -\tilde{e}_{31}\int_v \left[z\left(\frac{\partial \varphi}{\partial z}\delta\frac{\partial \phi_x}{\partial x} \right) \right] dv \qquad K_{44} = -\tilde{\varepsilon}_{zz}^s\int_v \left[\left(\frac{\partial \varphi}{\partial z}\delta\frac{\partial \varphi}{\partial z} \right) \right] dv$$

$$K_{45} = -\frac{1}{2}\bar{P}_z\int_v \left(\frac{\partial \varphi}{\partial z}\delta\theta \right) dv \qquad K_{51} = -\frac{1}{2}\bar{\beta}_{xx}\int_v \left(\theta^T\delta\frac{\partial u_0}{\partial x} \right) dv$$

$$K_{53} = -\frac{1}{2}\bar{\beta}_{xx}\int_v \left[\left(\theta\delta\frac{\partial \phi_x}{\partial x} \right) \right] dv \qquad K_{54} = -\frac{1}{2}\bar{P}_z\int_v \left(\theta\delta\frac{\partial \varphi}{\partial z} \right) dv$$

where; A_{ij}, B_{ij}, and D_{ij} are the laminate extensional, coupling, and bending stiffness coefficients and they are given by:

$$\left(A_{11}, B_{11}, D_{11} \right) = \sum_{n=1}^{N}\int_{-h/2}^{h/2}\tilde{Q}_{11}\left(1, z, z^2 \right) dz$$

and

$$A_{55} = \int_{-h/2}^{h/2}\left(\tilde{Q}_{55} \right) dz \tag{51}$$

The first variation of the kinetic energy Eqn. (25) is expressed as:

$$\delta T_e = \int_v \rho\left[\dot{u}\,\delta\dot{u} + \dot{w}\,\delta\dot{w} \right] dv \tag{52}$$

From equation (52) the elements of the mass matrix are given as:

$$M_{11} = I_\circ\int_0^L \left(\delta u_o^T u_o \right) dx \qquad M_{12} = M_{21} = 0 \tag{53}$$

$$M_{13} = -I_1 \int_0^L \left(\delta u_o^T \phi_x \right) dx \qquad M_{22} = I_\circ \int_0^L \left(\delta w_o^T w_\circ \right) dx$$

$$M_{23} = M_{32} = 0 \qquad M_{31} = -I_1 \int_0^L \left(\delta \phi_x^T u_\circ \right) dx$$

$$M_{33} = I_2 \int_0^L \left(\delta \phi_x^T \phi_x \right) dx$$

and $(I_0, I_1, I_2) = \rho \int_A (1, z, z^2) dA$

where, I and ρ is the mass moment of inertia and the mass density of the material, respectively.

The first variation of the heat energy, Eqn. (30), is expressed as:

$$\delta U_c = \int_v \left(k_{xx} \frac{dT}{dx} \delta \frac{dT}{dx} \right) dv \tag{54}$$

The first variation of the external work, Eqn. (33), is given by:

$$\delta W_{Total} = \int_0^L f_a \delta u dx + \int_0^L f_t \delta w dx + P_i \delta w_i - \int_{S_1} \sigma \delta \varphi dS_1 - \int_v \bar{Q} T\, \delta T dv - \int_{S_2} q^* T\, \delta T dS - \int_{S_3} h_c T_\infty \delta T dS + \int_{S_3} h_c T\, \delta T dS \tag{55}$$

By performing the integration of Eqn. (55), the first five terms which represent the mechanical and electrical loads are obtained and given by [29, 31]. The next three terms can be expressed as:

$$f_Q = \frac{\bar{Q} A L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{56}$$

$$f_q = \frac{q^* P L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{57}$$

$$f_h = \frac{h_c T_\infty P L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{58}$$

where the parameter $dS = Pdx$, where P is the perimeter of the element and assumed to be constant. And A and L are the cross section area and the length of the beam element. The vector for the thermal load is expressed as:

$$[Q] = \frac{\bar{Q} A L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{q^* P L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{h_c T_\infty P L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{59}$$

The summation of the integrations of Eqn. (54) and the last term of Eqn. (55), give the thermal effect matrix which can be expressed as:

$$K_{\theta\theta} = \frac{A k_{xx}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h_c P L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{60}$$

By substituting equations (35), (37), (39), (45),and (47) into equations (50), (51), and Eqn. (55), and perform the integration of these equations for a beam element with length L, width b and height h, the element stiffness matrix, mass matrix, and mechanical, electrical, thermal loads vectors for the beam element are obtained.

Equation of Motion

The equation of motion of the whole structure system is expressed as follows:

$$[M]\{\ddot{U}\}+[K_{uu}]\{U\}+[K_{u\varphi}]\{\varphi\}+[K_{u\theta}]\{\theta\}=\{F\} \tag{61}$$

$$[K_{\varphi u}]\{U\}+[K_{\varphi\varphi}]\{\varphi\}+[K_{\varphi\theta}]\{\theta\}=\{G\} \tag{62}$$

From Eqns. (7) and (29) and for isentropic process for which $S=0$, thus the heat equation is expressed as:

$$[K_{\theta\theta}]\{\theta\}=\{Q\} \tag{63}$$

where, $\{U\}$ is the structural displacements vector, $\{\Phi\}$ is the generalized electric degrees of freedom, $\{\theta\}$ is the generalized temperature degrees of freedom vector. $\{F\}$, $\{G\}$, and $\{Q\}$ are the mechanical, electric, and thermal loads vectors, respectively.

Stress Formulation

The displacements of the system are defined by relate the displacements at any point to the nodal displacements through the assumed shape functions as follows:

$$\{\bar{u}\}=[f]\{q\} \tag{64}$$

where, $[f]$ is a square matrix of the shape functions and its diagonal is given by:

$$[diagf]=[\xi_1 \quad \zeta_1 \quad \psi_1 \quad \varsigma_1 \quad \eta_1 \quad \xi_2 \quad \zeta_2 \quad \xi_3 \quad \xi_4 \quad \zeta_3 \quad \psi_2 \quad \varsigma_2 \quad \eta_2] \tag{65}$$

and $\{q\}$ is the nodal displacements vector and given by:

$$\{q\}=\{u_1 \quad w_1 \quad \phi_{x_1} \quad \varphi_1 \quad \theta_1 \quad u_2 \quad w_2 \quad u_3 \quad u_4 \quad w_3 \quad \phi_{x_2} \quad \varphi_2 \quad \theta_2\}' \tag{66}$$

The strain displacement relations is given by:

$$\{\varepsilon\}=[d]\{\bar{u}\} \tag{67}$$

where the matrix $[d]$ is a linear differential operator expresses the strain vector $\{\varepsilon\}$ in terms of displacements vector $\{\bar{u}\}$. Substitution of Eqn. (64) into Eqn. (67) yields:

$$\{\varepsilon\}=[d][f]\{q\} \tag{68}$$

$$\{\varepsilon\}=[B]\{q\} \tag{69}$$

where, $[B]$ is the square matrix gives the strains at any point within the element due to unit value of nodal displacements and it has the shape function differentiation. The diagonal of the matrix $[B]$ is given by:

$$[diagB]=\begin{bmatrix}\xi_{1,x} & \zeta_{1,x} & \psi_{1,x} & \varsigma_{1,x} & \eta_{1,x} & \xi_{2,x} & \zeta_{2,x} & \xi_{3,x} & \xi_{4,x} & \zeta_{3,x} & \psi_{2,x} & \varsigma_{2,x} & \eta_{2,x}\end{bmatrix} \quad (70)$$

Substitute equation (69) into equation (14), the stress components σ_{xx} and σ_{xz} can be obtained.

VALIDATION EXAMPLES

A MATLAB code is constructed to perform the analysis of graphite epoxy beam with piezoelectric actuators. The model inputs are the geometric and the material properties for piezoelectric and substrate materials, also the applied electrical, thermal, and mechanical loads values. The model is capable for predicting the nodal displacements, normal and shear stresses of the beam system. A set of examples are considered in this studies.

Example 1

A beam with material and geometric properties of piezoelectric composite beam are given in Table 1 and shown in Figure 2. The considered graphite epoxy beam is a $[0_8/p]$ where p is the piezoelectric layer, subjected to uniform thermal loads of T = 100 ^{0}C. The presence of the piezoelectric layers results in an asymmetric laminate configuration, which induced thermal distortions under the thermal load. The obtained results are compared with the others obtained by Ref. [4].

Figure 3 illustrate the transverse displacements produced when the beam is operating in closed circuit conditions. The closed circuit condition is obtained by grounding (0 V) both the lower and upper surfaces of the piezoelectric layer. Grounding both surfaces effectively eliminates the induced piezoelectric strain and produces a virtually conventional composite beam.

Example 2

A laminated composite beam with fibers orientations angles $[PZT\ /(0^0/45^0/-45^0/90^0)_s]$ and with material and geometric properties given in Table 1 is used for model validation. The beam is subjected to different values of thermal load, electrical load, and a mechanical load with intensity P=1N in some cases and equal zero in other cases. The beam has length to height ratio of (L/h=10). Figures 4 and 5 show the normal and shear stress distributions through the thickness of the beam for T=200 and V=500 volt. In addition, Figures 6 and 7 show the same stresses for a values of T = 100-500 ^{0}C and V=400 volt.

Table 1. Geometric and material properties of smart beams [4].

Properties	Graphite/epoxy	Piezoelectric APC 840
Elastic modulus E_{11} GPa $(Mpsi)$	39 (5.7)	68 (10.0)
Elastic modulus E_{22} GPa $(Mpsi)$	8.6 (1.24)	68 (10.0)
Shear modulus, G_{12}, GPa $(Mpsi)$	3.8 (0.54)	26.2 (3.84)
Major Poisson's ratio, ν_{12}	0.28	0.30
Minor Poisson's ratio, ν_{21}	0.06	0.30
Density $\rho, g/cm^3$ (lb/in^3)	2.1(0.076)	7.6 (0.27)
Thermal expansion coefficient $\alpha_{11}, 10^{-6}/^0C(10^{-6}/^0F)$	7.0 (3.9)	3.8 (2.1)
Thermal expansion coefficient $\alpha_{22}, 10^{-6}/^0C(10^{-6}/^0F)$	21.0 (11.7)	3.8 (2.1)
Piezoelectric charge constant $d_{31}, 10^{-12} m/V$	0	-125
Electric permittivity ε_{33} $10^{-9} N/V^2$	0	11.06
Pyro-electric constant p_3 $10^{-3} C/(m^{20}C)$	0	-0.25
Reference temperature $T_0, ^0C$ (^0F)	20.0 (68.0)	20.0 (68.0)
Beam length L cm (in)	25.4 (10)	25..4(10)
Beam wide cm (in)	2.54 (1.0)	2.54(1.0)
Layer thickness t cm (in)	0.0127(0.005)	0.0127(0.005)

CONCLUSIONS

A finite element model was proposed to predict deformation and stress response of beams subjected to thermo-electro-mechanical loads using first-order shear deformation theory was proposed. The following conclusions have been drawn:

1. The good agreement between the present model predictions using Timoshenko beam theory, and the corresponding results of other investigator proves the predictive capabilities of the present thermo-electro-mechanical model.
2. The resultant thermal deformations of the piezoelectric and the internal stress state of the beams with and without application of the voltage were investigated.
3. As the applied temperature increases, with certain value of applied voltage, both the beam deformation and internal stress values were increased, respectively.
4. The obtained results from the proposed finite element model were obtained at

resizable number of elements.

5. The model can be extended to model the a different cases of heat source and thermal radiation.

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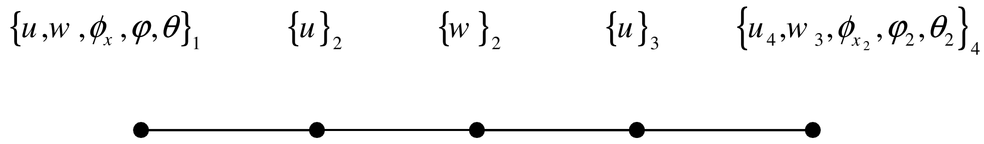

Fig. 1. Element nodal degrees of freedom.

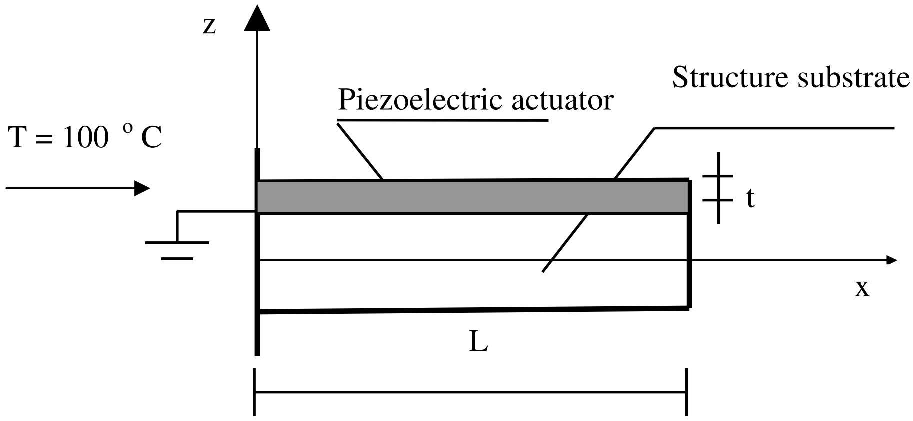

Fig. 2. Piezoelectric composite cantilever beam subjected to thermal load (T=100° C).

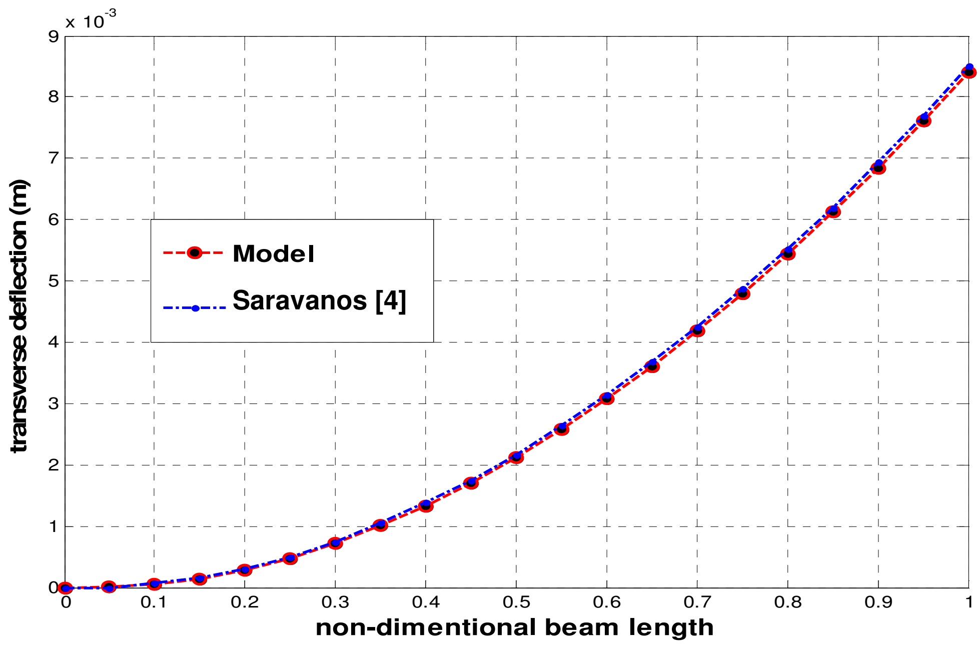

Fig. 3. Transverse deflection of present model compared with Ref. [4].

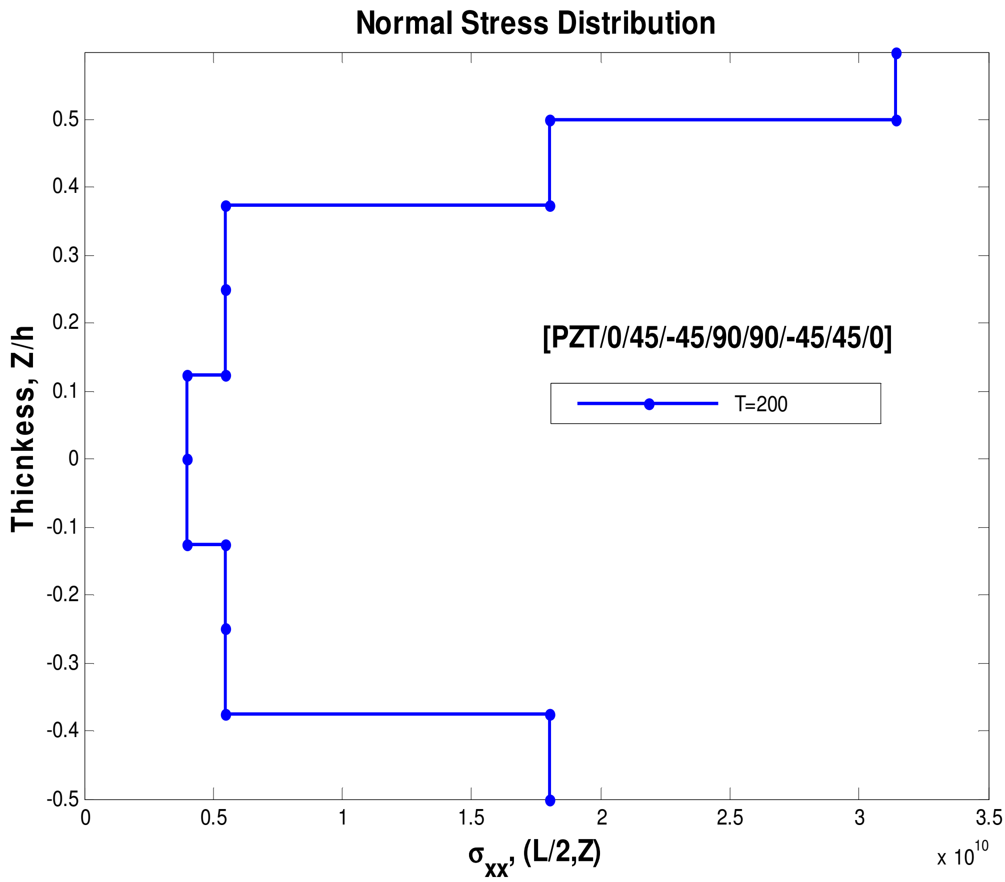

Fig. 4. Normal stress distribution for piezoelectric composite beam [PZT/(0^o/45^o/-45^o/90^o)$_s$], T=200^oC, and V=500 volt.

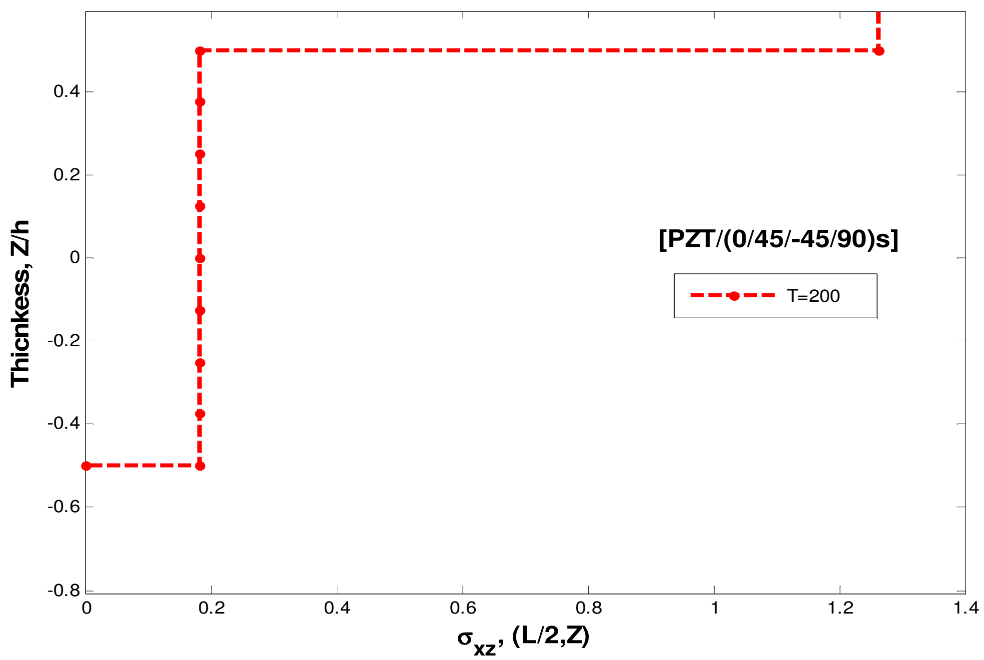

Fig. 5. Shear stress distribution for piezoelectric composite beam PZT/(0^o/45^o/-45^o/90^o)$_s$], T=200^oC, and V=500 volt.

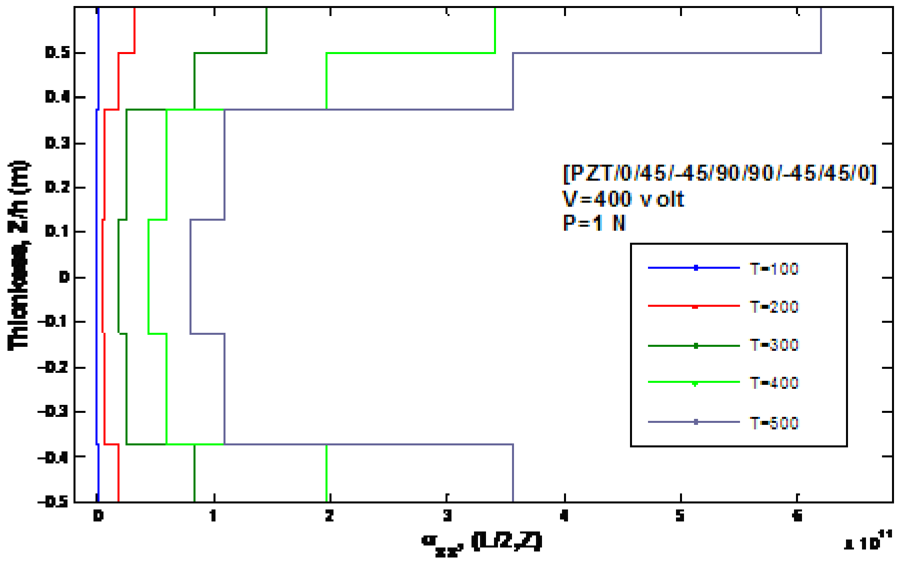

Fig. 6. Normal stress distribution for piezoelectric composite beam PZT/(0°/45°/-45°/90°)$_s$], P=1 N,T=100-500°C, and V=400 volt.

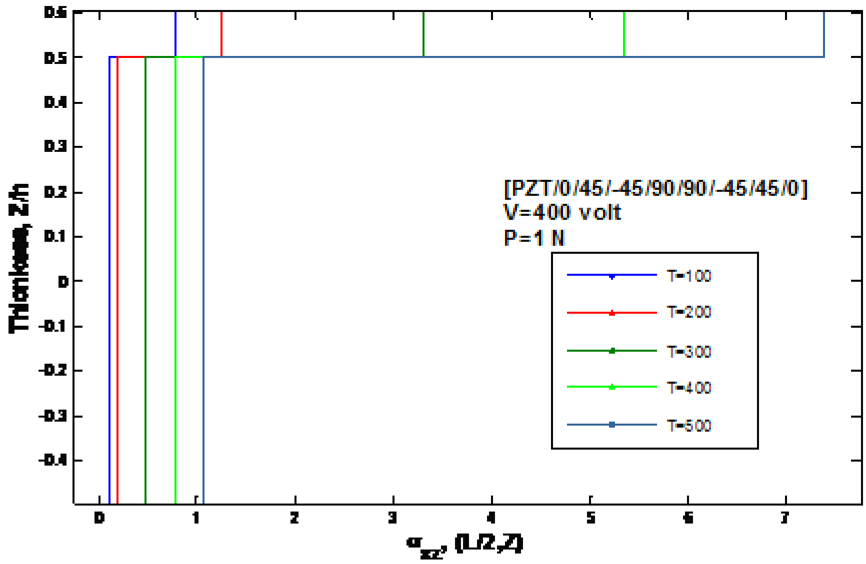

Fig. 7. Shear stress distribution for piezoelectric composite beam PZT/(0°/45°/-45°/90°)$_s$], P=1 N,T=100-500°C, and V=400 volt.