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TRANSVERSE OSCILLATION OF THE RAIL WHICH LIES ON THE DISCRETE BASE

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ABSTRACT

Friction was neglected by many authors when modeling the transverse oscillation of the rail during its movement. The non-considered contact friction leads to a violation of the law of conservation of momentum. In such a case, all the energy spent on the rotation of the railway wheels around the railway train axes and composition did not make a move. This problem definition was solved during movement of one wheel.

In this paper, the analytical solution of the problem of transverse oscillation of the rail which lies on the discrete base during movement of the wheel with taking into account dry friction on the «wheel-rail» contact is obtained. Research of transverse oscillation of the rail during its movement considering rails junctions, number of ties, railway train axes, space and reflected waves propagation from the rails' wheels and butts is conducted.

KEY WORDS

Railway transport, dry friction, «wheel-rail» contact, rail junctions, mathematical model, motion dynamics, wave theory.

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NOMENCLATURE

t	Time
m	Mass of unit of the rail length
x	Abscissa of the current rail section
x_j	Coordinates of the *j*-th tie
s	Number of ties
$y(x,t)$	Rail deflection
c	Base stiffness
E	Modulus of elasticity of rail
J	Moment of inertia of the rail cross-section
v_0	Movement rate of wheel on the rail
τ_k	Force of rolling dry friction
$\delta(v_0 t - x)$	Dirac delta function
l	Rail length

INTRODUCTION

At the present time, there are a large number of works [1-3] which describe the interaction of the rail with movable loads. However, these papers consider rail which lies on a continuous elastic base and without dry friction on the «wheel-rail» contact. Taking into account discrete base of the rail leads to the determination of the set of unknown reactions of discrete connections by the number which is equal to the number of sleepers, that represents a significant difficulty to obtain a decision even in statics. Taking into account such a nonlinear dissipation mechanism as dry friction between the wheels and the rail, generates additional complications problems (Figure 1). If dry friction is not considered, it violates the law of conservation of momentum in classical mechanics. This means that when the power is turned on and its further growth, the composition is standstill and is not capable of moving. All the power must go to the wheel with increasing angular velocity revolved around its axis; i.e. there is no translational motion (skidding system).

However, in practice, the movement takes place, it means that the friction, which is not counted in theory, takes place in practice and provides a necessary movement of the composition. Dry friction is not an active force, but at the same time provides the movement of rail transport. Research in this field in the world held for about two centuries , meanwhile, there is no precedent inclusion of this force in the equation of motion .

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In this problem, ties are taken as discrete elastic connections. Then the differential equation of motion is described as follows:

$$EJ\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} + c\sum_{j=0}^{s}[1-\delta(x-x_j)]y = -\tau_k \delta(x-\upsilon_0 t), \quad (1)$$

Equation (1) is reduced to the following form:

$$\frac{\partial^4 y}{\partial x^4}+\frac{m}{EJ}\frac{\partial^2 y}{\partial t^2}+\alpha\sum_{j=0}^{s}[1-\delta(x-x_j)]y=-\frac{\tau_k}{EJ}\delta(x-v_o t), \tag{2}$$

where $$\alpha=\frac{c}{EJ} \tag{3}$$

Initial conditions:

when $$t=0: \quad y(x,0)=0, \quad \frac{\partial y(x,0)}{\partial t}=0. \tag{4}$$

Boundary conditions:

when $$x=0: \quad y(0,t)=0, \quad \frac{\partial^2 y(0,t)}{\partial x^2}=0. \tag{5}$$

when $$x=l: \quad y(l,t)=0, \quad \frac{\partial^2 y(l,t)}{\partial x^2}=0. \tag{6}$$

Applying the Tyurekhodjaev's method of partial discretization of nonlinear equations [4] to equation (2) with a discontinuous right-hand side we get the following result:

$$\begin{aligned}\frac{\partial^4 y}{\partial x^4}+\frac{m}{EJ}\frac{\partial^2 y}{\partial t^2}&=-\frac{\alpha}{2}\sum_{k=1}^{n}\sum_{j=0}^{s}(x_k+x_{k+1})[1-\delta(x-x_j)]\times\\&\times[u_k\delta(x-x_k)-u_{k+1}\delta(x-x_{k+1})]-\frac{\tau_k}{EJ}\delta(x-v_0 t),.\end{aligned} \tag{7}$$

where x_j is the coordinates of the discretization points.

Consider the homogeneous equation:

$$\frac{\partial^4 y}{\partial x^4}+\frac{m}{EJ}\frac{\partial^2 y}{\partial t^2}=0 \tag{8}$$

The solution of the equation of motion, which corresponds to eigen oscillations, is presented as [5]:

$$y(x,t)=u(x)\cos(pt+\varphi), \tag{9}$$

where $u(x)$ is the amplitude function, p is the angular frequency of oscillation.

Substituting by Eqn. (9) into Eqn. (8), we obtain the following ordinary differential equation:

$$\frac{d^4u}{dx^4} - a^4u = 0, \tag{10}$$

where
$$a^4 = \frac{p^2m}{EJ}. \tag{11}$$

Roots of the characteristic equation corresponding to equation (10) will be $\pm a$ and $\pm a_i$. In accordance with it, the solutions of the homogeneous equation (10) can be expressed in terms of ax trigonometric and exponential functions:

$$u = C_1e^{ax} + C_2e^{-ax} + C_3\cos(ax) + C_4\sin(ax). \tag{12}$$

Taking into account Eqn. (9), equation (7) is written in the following form:

$$\begin{aligned}\frac{d^4u}{dx^4} - a^4u = -\frac{\alpha}{2}\sum_{k=1}^{n}\sum_{j=0}^{s}(x_k + x_{k+1})\left[1 - \delta(x - x_j)\right]\times \\ \times\left[u_k\delta(x - x_k) - u_{k+1}\delta(x - x_{k+1})\right] - \frac{\tau_k}{EJ}\delta(x - v_0t)\frac{1}{\cos(pt+\varphi)}.\end{aligned} \tag{13}$$

The right-hand side of Eqn. (13) is denoted as:

$$\begin{aligned}\Phi(x,t) = -\frac{\alpha}{2}\sum_{k=1}^{n}\sum_{j=0}^{s}(x_k + x_{k+1})\{u_k\left[\delta(x - x_k) - \delta(x_k - x_j)\right] - \\ - u_{k+1}\left[\delta(x - x_{k+1}) - \delta(x_{k+1} - x_j)\right]\} - \frac{\tau_k}{EJ}\delta(x - v_0t)\frac{1}{\cos(pt+\varphi)}.\end{aligned} \tag{14}$$

Lagrange's method of variation of undetermined coefficients is used to obtain the general solution of the non-homogeneous differential equation (13) as follows:

$$\begin{aligned}u(x) = {} & C_1e^{ax} + C_2e^{-ax} + C_3\cos(ax) + C_4\sin(ax) - \\ & -\frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}(x_k + x_{k+1})\times \\ & \times\{u_k\left[\left[sh(a(x - x_k)) - \sin(a(x - x_k))\right]H(x - x_k) + 3\delta(x_k - x_j)\right] - \\ & - u_{k+1}\left[\left[sh(a(x - x_{k+1})) - \sin(a(x - x_{k+1}))\right]H(x - x_{k+1}) + 3\delta(x_{k+1} - x_j)\right]\} + \\ & + \frac{2\tau_k}{3a^4EJ\cos(pt+\varphi)}\left[ch(a(x - v_0t)) + \sin(a(x - v_0t))\right]H(x - v_0t).\end{aligned} \tag{15}$$

Substituting by Eqns. (9) and (15) into Eqn. (7), the general solution of the non-homogeneous differential equation, i.e. Eqn. (7), is represented by:

$$y(x,t)=\left[C_1e^{ax}+C_2e^{-ax}+C_3\cos(ax)+C_4\sin(ax)\right]\cos(pt+\varphi)-\frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}\left(x_k+x_{k+1}\right)\times$$
$$\times\left\{y_k.\left[\left[sh\left(a\left(x-x_k\right)\right)-\sin\left(a\left(x-x_k\right)\right)\right]H\left(x-x_k\right)+3\delta\left(x_k-x_j\right)\right]-\right.$$
$$\left.-y_{k+1}.\left[\left[sh\left(a\left(x-x_{k+1}\right)\right)-\sin\left(a\left(x-x_{k+1}\right)\right)\right]H\left(x-x_{k+1}\right)+3\delta\left(x_{k+1}-x_j\right)\right]\right\}+ \tag{16}$$
$$+\frac{2\tau_k}{3a^4EJ}\left[ch\left(a\left(x-v_0t\right)\right)+\sin\left(a\left(x-v_0t\right)\right)\right]H\left(x-v_0t\right).$$

C_1, C_2, C_3 and C_4 are determined using boundary conditions, i.e. Eqns. (5) and (6):

$$C_1=\frac{-\left(F_1+F_2\right)e^{-al}+\left(F_3+F_4\right)}{4sh(al)},$$
$$C_2=\frac{\left(F_1+F_2\right)e^{al}-\left(F_3+F_4\right)}{4sh(al)},$$
$$C_3=\frac{F_1-F_2}{2}\text{, and} \tag{17}$$
$$C_4=-\frac{\left(F_1-F_2\right)\cos(al)+\left(F_3-F_4\right)}{2\sin(al)}.$$

where:

$$F_1=\frac{1}{\cos(pt+\varphi)}\left\{\frac{\alpha}{a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}\left(x_k+x_{k+1}\right)\left[y_k\delta\left(x_k-x_j\right)-y_{k+1}\delta\left(x_{k+1}-x_j\right)\right]-\right.$$
$$\left.-\frac{2\tau_k}{3a^4EJv_0}\left[ch\left(a\left(-av_0t\right)\right)-\sin\left(a\left(av_0t\right)\right)\right]\delta(t)\right\}, \tag{18}$$

$$F_2=\frac{-2\tau_k}{3a^5EJv_0\cos(pt+\varphi)}\left\{a\left[ch\left(av_0t\right)+\sin\left(av_0t\right)\right]+2\left[-sh\left(av_0t\right)+\cos\left(av_0t\right)\right]\delta(t)\right., \tag{19}$$

$$F_3=\frac{1}{\cos(pt+\varphi)}\left\{\frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}\left(x_k+x_{k+1}\right)\times\right.$$
$$\times\left\{y_k\left[\left[sh\left(a\left(l-x_k\right)\right)-\sin\left(a\left(l-x_k\right)\right)\right]H\left(l-x_k\right)+3\delta\left(x_k-x_j\right)\right]-\right.$$
$$\left.-y_{k+1}\left[\left[sh\left(a\left(l-x_{k+1}\right)\right)-\sin\left(a\left(l-x_{k+1}\right)\right)\right]H\left(l-x_{k+1}\right)+3\delta\left(x_{k+1}-x_j\right)\right]\right\}- \tag{20}$$
$$-\frac{2\tau_k}{3a^4EJ}\left[ch\left(a\left(l-v_0t\right)\right)+\sin\left(a\left(l-v_0t\right)\right)\right]H\left(l-v_0t\right)$$

and

$$
\begin{aligned}
F_4 = \frac{1}{a^2\cos(pt+\varphi)} & \left\{ \frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}\left(x_k + x_{k+1}\right)\times \right. \\
& \times\{y_k\left[a^2\left[sh(a(l-x_k)) + \sin(a(l-x_k))\right]H(l-x_k) + \right. \\
& + 2a\left[ch(a(l-x_k)) - \cos(a(l-x_k))\right]\delta(l-x_k) + \\
& + \left[sh(a(l-x_k)) - \sin(a(l-x_k))\delta'(l-x_k)\right] - \\
& - y_{k+1}\left[a^2\left[sh(a(l-x_{k+1})) + \sin(a(l-x_{k+1}))\right]H(l-x_{k+1}) + \right. \\
& + 2a\left[ch(a(l-x_{k+1})) - \cos(a(l-x_{k+1}))\right]\delta(l-x_{k+1}) + \\
& + \left[sh(a(l-x_{k+1})) - \sin(a(l-x_{k+1}))\right]\delta'(l-x_{k+1})\left.\right]\} - \\
& - \frac{2\tau_k}{3a^4 EJ}\left\{a^2\left[ch(a(l-v_0 t)) - \sin(a(l-v_0 t))\right]H(l-v_0 t) + \right. \\
& + 2a\left[sh(a(l-v_0 t)) + \cos(a(l-v_0 t))\right]\delta(l-v_0 t) + \\
& + \left[ch(a(l-v_0 t)) + \sin(a(l-v_0 t))\right]\delta'(l-x_k)\left.\right\}\left.\right\}.
\end{aligned} \tag{21}
$$

Substituting by Eqn. (17) into Eqn. (16), the solution of the differential equation of motion of the transverse oscillations of the railway which lies on discrete elastic base is:

$$
\begin{aligned}
y(x,t) = & \left[\frac{-(F_1+F_2)e^{-al} + (F_3+F_4)}{4sh(al)}e^{ax} + \frac{(F_1+F_2)e^{al} - (F_3+F_4)}{4sh(al)}e^{-ax} + \right. \\
& \left. + \frac{(F_1-F_2)}{2}\cos(ax) - \frac{(F_1-F_2)\cos(al) + (F_3-F_4)}{2\sin(al)}\sin(ax)\right]\cos(pt+\varphi) - \\
& - \frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}\left(x_k + x_{k+1}\right)\times \\
& \times\{y_k\left[sh(a(x-x_k)) - \sin(a(x-x_k))\right]H(x-x_k) + 3\delta(x_k - x_j) - \\
& - y_{k+1}\left[sh(a(x-x_{k+1})) - \sin(a(x-x_{k+1}))\right]H(x-x_{k+1}) + 3\delta(x_{k+1} - x_j)\} + \\
& + \frac{2\tau_k}{3a^4 EJ}\left[ch(a(x-v_0 t)) + \sin(a(x-v_0 t))\right]H(x-v_0 t).
\end{aligned} \tag{22}
$$

When $k \geq 1$ and time $t \neq 0 \Rightarrow \delta(t) = 0$, therefore $F_1 = 0$, $F_2 = 0$ and C_1, C_2, C_3 and C_4 take the following values:

$$
C_1 = \frac{F_3+F_4}{4sh(al)},\quad C_2 = \frac{-F_3+F_4}{4sh(al)},\quad C_3 = 0,\quad C_4 = \frac{F_3-F_4}{2\sin(al)}. \tag{23}
$$

Taking into account Eqn. (23), equation (22) after some transformations leads to the following form:

$$y(x,t)=\frac{\cos(pt+\varphi)}{sh(al)\sin(al)}\{F_3[\sin(al)sh(ax)-sh(al)\sin(ax)]+F_4[\sin(al)sh(ax)+sh(al)\sin(ax)]\}-$$

$$-\frac{\alpha}{3a^4}\sum_{k=1}^{n}\sum_{j=0}^{s}(x_k+x_{k+1})\times$$

$$\times\{y_k[[sh(a(x-x_k))-\sin(a(x-x_k))]H(x-x_k)+3\delta(x_k-x_j)]- \tag{24}$$

$$-y_{k+1}[[sh(a(x-x_{k+1}))-\sin(a(x-x_{k+1}))]H(x-x_{k+1})+3\delta(x_{k+1}-x_j)]\ \}+$$

$$+\frac{2\tau_k}{3a^4EJ}[ch(a(x-v_0t))+\sin(a(x-v_0t))]H(x-v_0t).$$

CONCLUSION

The above formulation of the problem of transverse deformation of the rail train, which lies on the discrete elastic base during movement of the wheel with taking into account dry friction on the «wheel-rail» contact, is performed for the first time. The analytical solution of oscillation of the rail is obtained.

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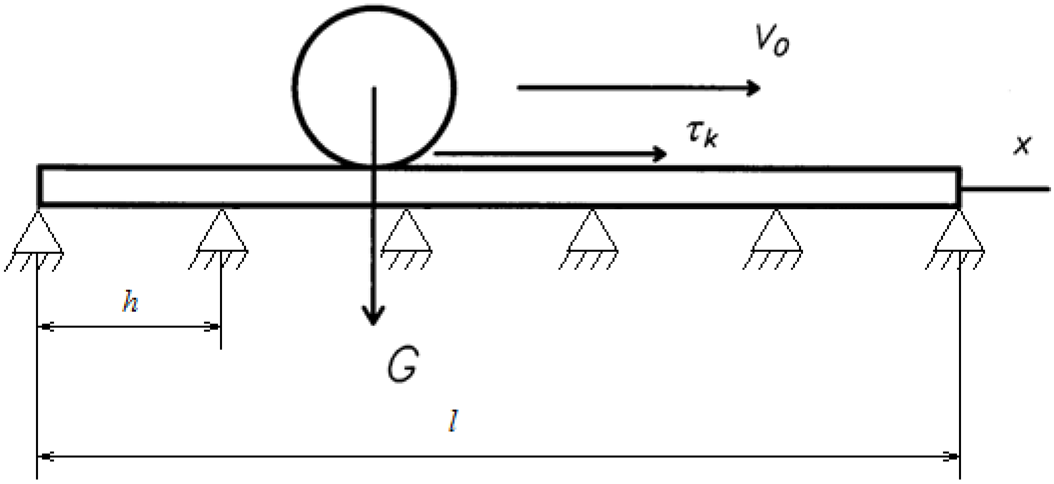

Fig. 1. A schematic drawing for rolling wheel on the rail which lies on the discrete base